Linear inviscid damping and vorticity depletion for shear flows

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The incompressible Euler equations

We consider the incompressible Euler equations in $\mathbb{R}^+ \times \Omega \subseteq \mathbb{R}^{d+1}$, which take as follows:

$$(E) \quad \begin{cases} \partial_t V + V \cdot \nabla V + \nabla P = 0, \\ \nabla \cdot V = 0, \\ V \cdot n = 0 \quad \text{on} \quad \partial\Omega, \\ V(0, x) = V_0(x). \end{cases}$$

Here $V = (V^1, \cdots, v^d)$ is the velocity and P is the pressure.

Classical results

Classical results on the well-posedness of (E):

- Kato-Ponce theorem(CPAM 1988):
  local well-posedness(LWP) for $V_0 \in H^s(\mathbb{R}^d), s > 1 + \frac{d}{2}$.
- Beale-Kato-Majda break-down criterion(CMP 1984):
  maximal existence time $T^* < +\infty$ if and only if

$$\int_0^{T^*} \|\omega(t)\|_{L^\infty} dt = +\infty, \quad \omega = \nabla \times V.$$

- Bourgain-Li theorem(Inven Math 2015):
  Ill-posedness in the critical space $H^{1+\frac{d}{2}}(\Omega)$ for $\Omega = \mathbb{R}^d, \mathbb{T}^d$.

GWP for 2-D Euler equations

- Let $\omega = \partial_x V^2 - \partial_y V^1$ be the vorticity, which satisfies

$$\omega_t + V \cdot \nabla \omega = 0.$$

- Maximum principle:

$$\|\omega(t)\|_{L^\infty} \leq \|\omega_0\|_{L^\infty},$$

which in particular implies GWP for 2-D Euler equations.

- **Many interesting open questions**: Large time behaviour of the solution, the self-organization into coherent structures?

Stability of coherent structures

The stability of coherent structures has been an active field of fluid mechanics, which started in the nineteenth century with Rayleigh, Kelvin, Orr, Sommerfeld and many others. Let us mention some of classical results: Rayleigh's inflection theorem, Howard's semicircle theorem, and Arnold's criterion for Lyapunov stability.

We are concerned with the asymptotic stability of the 2-D Euler equations around the shear flow $(u(y), 0)$ in a finite channel, which is a steady solution of (E) in a channel. The first step toward this problem is to study the linearized Euler equations around $(u(y), 0)$.

The linearized Euler equations

- $\Omega = \mathbb{T} \times I$ is a finite channel.
- Shear flow

$$V(x, y) = (u(y), 0).$$

- The linearized Euler equation around shear flow:

$$\partial_t \omega + \mathcal{L}\omega = 0, \quad \omega|_{t=0} = \omega_0(x, y),$$

where $\mathcal{L} = u(y)\partial_x + u''(y)\partial_x(-\Delta)^{-1}$.

Linear damping for couette flow

- Linear vorticity equation around couette flow $(y, 0)$

$$\omega_t + y\partial_x\omega = 0 \Rightarrow \omega(t, x, y) = \omega_0(x - ty, y).$$

- $\Omega = \mathbb{T} \times \mathbb{R}$(Orr 1907): let ψ be the stream function

$$\widehat{\psi}(t, \alpha, \xi) = \frac{\widehat{\omega}_0(\alpha, \xi + \alpha t)}{\alpha^2 + |\xi|^2},$$

  which implies that V^1 decays at t^{-1}, while V^2 decays at t^{-2}.

- Lin and Zeng(ARMA 2011): if $\int_{\mathbb{T}} \omega_0(x, y)dx = 0$, then
  1. if $\omega_0(x, y) \in H_x^{-1}H_y^1$, then $\|V(t)\|_{L^2} = O\left(\frac{1}{t}\right)$,
  2. if $\omega_0(x, y) \in H_x^{-1}H_y^2$, then $\|V^2(t)\|_{L^2} = O\left(\frac{1}{t^2}\right)$.

Landau damping

- Vlasov-Poisson(VP) equation

$$\partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = 0,$$

where $F = -eE$ with

$$E = \nabla(-\Delta)^{-1}\Big(4\pi(\rho - \bar{\rho})\Big), \quad \bar{\rho} = \int_{\mathbb{R}^d} f(t,x,v)dv.$$

- The linearized VP equation around $f_0(v)$:

$$\partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f_0 = 0.$$

- Linear Landau damping(1946): $E(t,x)$ decays to zero.
- Nonlinear Landau damping: Mouhot and Villani(Acta Math 2011).

Nonlinear damping for couette flow

Due to possible nonlinear transient growth, it is a challenging task from linear damping to nonlinear damping.

- Bedrossian and Masmoudi(2013): nonlinear inviscid damping for the perturbations in Gevrey class in $\Omega = \mathbb{T} \times \mathbb{R}$(still open in in a finite channel).
- Lin and Zeng(ARMA 2013):

For any $T > 0$ and $0 \leq s < \frac{3}{2}$, then for any $\epsilon > 0$, there exists steady solution $(V_\epsilon^1, V_\epsilon^2)$ has a minimal x-period T so that

$$\|\omega_\epsilon - 1\|_{H^s} < \epsilon, \quad V_\epsilon^2(x, y) \neq 0.$$

This implies that nonlinear damping does not hold for the perturbation(vorticity) in H^s with $s < \frac{3}{2}$.

Linear damping for monotone shear flow

For general shear flow, the linear inviscid damping is also a difficult problem, which is associated with the singularities at the critical layers $u = c$ of the solution for the Rayleigh equation

$$(u - c)(\phi'' - \alpha^2\phi) - u''\phi = f.$$

In fact, Ran u is just the continuous spectrum of $\mathcal{L}$, whose properties and the non-normality of $\mathcal{L}$ are related to many important phenomena such as transient growth, inviscid damping and algebraic instabilities.

In 1960, Case gave a first prediction of linear damping for monotone shear flow based on the Laplace transform.

Linear damping for monotone shear flow

Some rigorous mathematical results:

- Rosencrans and Sattinger(1966) for analytic monotone shear flow:

$$\widehat{\psi}(t,\alpha,y) \sim t^{-1}.$$

- Stepin(1995) for monotone shear flow $u(y) \in C^{2+\mu_0}$ without inflection points:

$$\widehat{\psi}(t,\alpha,y) \sim t^{-\nu} \quad \nu < \mu_0.$$

- Zillinger(2014): linear damping under the assumptions

$$L\|u''\|_{W^{3,\infty}} \le \epsilon, \quad \omega_0(x,0) = 0.$$

Linear damping for monotone shear flow

Theorem 1.(Wei-Zhang-Zhao CPAM 2017):

Let $u(y) \in C^4([0,1])$ be a monotone function. Suppose that the linearized operator $\mathcal{L}$ has no embedding eigenvalues. Assume that $\int_{\mathbb{T}} \omega_0(x,y)dx = 0$ and $P_{\mathcal{L}}\omega_0 = 0$, where $P_{\mathcal{L}}$ is the spectral projection to $\sigma_d\big(\mathcal{L}\big)$. Then it holds that

1. if $\omega_0(x,y) \in H_x^{-1}H_y^1$, then
$$\|V(t)\|_{L^2} \le C\langle t\rangle^{-1}\|\omega_0\|_{H_x^{-1}H_y^1};$$
2. if $\omega_0(x,y) \in H_x^{-1}H_y^2$, then
$$\|V^2(t)\|_{L^2} \le C\langle t\rangle^{-2}\|\omega_0\|_{H_x^{-1}H_y^2};$$
3. Scattering in $H_x^{-1}H_y^k$ for $k = 0, 1$.

Linear damping for monotone shear flow

Some remarks on Theorem 1:

- If $u(y)$ has no inflection points, then $\mathcal{L}$ has no eigenvalues.
- If the wave-length L with respect to x is suitably small, then $\mathcal{L}$ has no embedding eigenvalues.
- Nonlinear damping: challenging open problem!

Linear damping for non-monotone shear flow

Two important examples: Poiseuille flow $u(y) = y^2$ and Kolmogorov flow $u(y) = \cos y$.

Bouchet and Morita's predictions based on Laplace tools and numerical computations (Physica D 2011):

- New dynamic phenomena: depletion phenomena of the vorticity at the stationary streamlines

$$\widehat{\omega}(t, \alpha, y) \sim \omega_\infty(y) \exp(-i\alpha u(y)t) + O(t^{-\gamma}),$$

where $\omega_\infty(y_c) = 0$ at stationary points y_c of $u(y)$.

- Decay estimates using stationary phase expansion.

We consider a class of shear flows denoted by $\mathcal{K}$, which consists of the function $u(y)$ satisfying $u(y) \in H^3(-1,1)$, and $u''(y) \neq 0$ for critical points(i.e., $u'(y) = 0$) and $u'(\pm 1) \neq 0$.

Theorem 2.(Wei-Zhang-Zhao arxiv 2017):

Assume that $u(y) \in \mathcal{K}$ and the linearized operator $\mathcal{R}_\alpha$ defined by (3) has no embedding eigenvalues. Assume that $\widehat{\omega}_0(\alpha, y) \in H^1_y(-1,1)$ and $P_{\mathcal{R}_\alpha}\widehat{\psi}_0(\alpha, y) = 0$, where ψ_0 is the stream function and $P_{\mathcal{R}_\alpha}$ is the spectral projection to $\sigma_d(\mathcal{R}_\alpha)$. Then it holds that

$$\|\widehat{V}(\cdot,\alpha,\cdot)\|_{L^2_t L^2_y} + \|\partial_t \widehat{V}(\cdot,\alpha,\cdot)\|_{L^2_t L^2_y} \leq C_\alpha \|\widehat{\omega}_0(\alpha,\cdot)\|_{H^1_y}.$$

In particular, $\lim\limits_{t\to+\infty} \|\widehat{V}(t,\alpha,\cdot)\|_{L^2_y} = 0.$

Linear damping for non-monotone shear flow

To obtain the explicit decay rate of the velocity, we consider a class of symmetric shear flow:

(S) $u(y) = u(-y)$, $u'(y) > 0$ for $y > 0$, $u'(0) = 0$ and $u''(0) > 0$.

An important example is the Poiseuille flow $u(y) = y^2$.

Theorem 3.(Wei-Zhang-Zhao arxiv 2017):

Assume that $u(y) \in C^4([-1,1])$ satisfies (S) with the same spectral assumption as in Theorem 1. Then it holds that

$$\|V(t)\|_{L^2} \le \langle t\rangle^{-1}\|\omega_0\|_{H_x^{-\frac{1}{2}}H_y^1}, \quad \|V^2(t)\|_{L^2} \le C\langle t\rangle^{-2}\|\omega_0\|_{H_x^{\frac{1}{2}}H_y^2}.$$

Moreover, if $\omega_0(x,y) \in H_x^{-\frac{1}{2}+k}H_y^k$ for $k = 0, 1$, there exists $\omega_\infty(x,y) \in H_x^{-\frac{1}{2}+k}H_y^k$ such that

$$\|\omega(t, x + tu(y), y) - \omega_\infty\|_{H_x^{-\frac{1}{2}+k}L_y^2} \longrightarrow 0 \quad \text{as} \quad t \to +\infty.$$

New dynamic phenomena: vorticity depletion

Let us first consider a passive transport equation

$$\partial_t \omega + u(y)\partial_x \omega = 0, \quad \omega(0,x,y) = \omega_0(x,y).$$

If $u'(y) > 0$ and $\int_{\mathbb{T}} \omega_0 dx = 0$, then we have

$$\|\omega(t)\|_{L^2_x H^{-1}_y} \le \frac{C}{\langle t \rangle} \|\omega_0\|_{H^{-1}_x H^1_y}. \tag{1}$$

If u satisfies (S) and $\int_{\mathbb{T}} \omega_0 dx = 0$, then we have

$$\|\omega(t)\|_{L^2_x H^{-1}_y} \le \frac{C}{\langle t \rangle^{\frac{1}{2}}} \|\omega_0\|_{H^{-\frac{1}{2}}_x H^1_y}. \tag{2}$$

New dynamic phenomena: vorticity depletion

The damping in (1) and (2) is due to vorticity mixing. So, this can not explain the enhanced damping in Theorem 3 when $u \in (S)$. In such case, the mechanism leading to the enhanced damping is the depletion of the vorticity at the stationary streamlines.

Theorem 4.(Wei-Zhang-Zhao arxiv 2017):

Under the same assumptions as in Theorem 2, if $u'(y_0) = 0$, then we have

$$\lim_{t \to +\infty} \widehat{\omega}(t, \alpha, y_0) = 0.$$

Reduction to the resolvent estimate

In terms of the stream function ψ, the linearized Euler equations take as follows

$$\partial_t \Delta\psi + u(y)\partial_x \Delta\psi - u''(y)\partial_x \psi = 0.$$

Taking the Fourier transform in x, we get

$$(\partial_y^2 - \alpha^2)\partial_t \widehat{\psi} = i\alpha\Big(u''(y) - u(y)(\partial_y^2 - \alpha^2)\Big)\widehat{\psi}.$$

Inverting the operator $(\partial_y^2 - \alpha^2)$, we get

$$-\frac{1}{i\alpha}\partial_t \widehat{\psi} = \mathcal{R}_\alpha \widehat{\psi},$$

where

$$\mathcal{R}_\alpha \widehat{\psi} = -(\partial_y^2 - \alpha^2)^{-1}\Big(u''(y) - u(\partial_y^2 - \alpha^2)\Big)\widehat{\psi}. \tag{3}$$

Reduction to the resolvent estimate

Classical results on the spectrum $\sigma(\mathcal{R}_\alpha)$:

1. The spectrum $\sigma(\mathcal{R}_\alpha)$ is compact;
2. The continuous spectrum $\sigma_c(\mathcal{R}_\alpha)$ is contained in the range $Ran(u)$ of $u(y)$;
3. The eigenvalues of $\mathcal{R}_\alpha$ can not cluster except possibly along on $Ran(u)$;
4. If $u(y)$ has no infection points in $[0,1]$, then $\mathcal{R}_\alpha$ has no embedding eigenvalues.

Then we have

$$\widehat{\psi}(t,\alpha,y) = \frac{1}{2\pi i}\int_{\partial\Omega} e^{-i\alpha tc}(c-\mathcal{R}_\alpha)^{-1}\widehat{\psi}(0,\alpha,y)dc,$$

where Ω is a simple connected domain including the spectrum $\sigma(\mathcal{R}_\alpha)$ of $\mathcal{R}_\alpha$.

Reduction to the resolvent estimate

Let Φ be a solution of the inhomogeneous Rayleigh equation

$$\begin{cases} \Phi'' - \alpha^2\Phi - \dfrac{u''}{u-c}\Phi = f, \\ \Phi(0) = \Phi(1) = 0. \end{cases} \tag{4}$$

with $f(\alpha, y, c) = \frac{\widehat{\omega}_0(\alpha,y)}{i\alpha(u-c)}$. It holds that

$$(c - \mathcal{R}_\alpha)^{-1}\widehat{\psi}(0, \alpha, y) = i\alpha\Phi(\alpha, y, c).$$

Thus, we obtain

$$\widehat{\psi}(t, \alpha, y) = \frac{1}{2\pi}\int_{\partial\Omega} \alpha\Phi(\alpha, y, c)e^{-i\alpha ct}dc.$$

Reduction to the resolvent estimate

Under the spectral assumption, it suffices to consider the complex constant $c \in \Omega_{\epsilon_0} \triangleq D_0 \cup D_{\epsilon_0} \cup B^l_{\epsilon_0} \cup B^r_{\epsilon_0}$, where

$$
\begin{aligned}
D_0 &\triangleq \Big\{c \in [u(0), u(1)]\Big\},\\
D_{\epsilon_0} &\triangleq \Big\{c = c_r + i\epsilon,\ c_r \in [u(0), u(1)], 0 < |\epsilon| < \epsilon_0\Big\},\\
B^l_{\epsilon_0} &\triangleq \Big\{c = u(0) + \epsilon e^{i\theta},\ 0 < \epsilon < \epsilon_0,\ \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\Big\},\\
B^r_{\epsilon_0} &\triangleq \Big\{c = u(1) - \epsilon e^{i\theta},\ 0 < \epsilon < \epsilon_0,\ \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\Big\}.
\end{aligned}
$$

Reduction to the resolvent estimate

Two key ingredients:

- The limiting absorption principle: as $\varepsilon \to 0$,

$$\Phi(\alpha, y, c \pm i\varepsilon) \to \Phi_\pm(\alpha, y, c) \quad \text{for } c \in \operatorname{Ran} u.$$

  The difficulty is that $\frac{1}{u(y)-c}$ is more singular if u has critical points. Bouchet and Morita's predictions are based on the hypothesis of the limiting absorption principle, which was verified by using numerical computations.
- Regularity of the limit $\Phi_\pm(\alpha, y, c)$?

The limiting absorption principle

Consider the inhomogeneous Rayleigh equation when $u \in \mathcal{K}$:

$$(u - c)(\Phi'' - \alpha^2 \Phi) - u''\Phi = \omega, \quad \Phi(-1) = \Phi(1) = 0. \qquad (5)$$

Using blow-up technique and compactness argument, we prove

Proposition. If $\mathcal{R}_\alpha$ has no embedding eigenvalues, then there exists ϵ_0 such that for $c \in \Omega_{\epsilon_0} \setminus D_0$, the solution to (5) has the the following uniform bound

$$\|\Phi\|_{H^1(-1,1)} \le C\|\omega\|_{H^1(-1,1)}.$$

Moreover, there exists $\Phi_\pm(\alpha, y, c) \in H_0^1(-1,1)$ for $c \in \mathrm{Ran}\, u$, so that $\Phi(\alpha, \cdot, c \pm i\epsilon) \to \Phi_\pm(\alpha, \cdot, c)$ in $C([-1,1])$ as $\epsilon \to 0+$ and

$$\|\Phi_\pm(\alpha, \cdot, c)\|_{H^1(-1,1)} \le C\|\omega\|_{H^1(-1,1)}.$$

Linear damping and vorticity depletion

Using the limiting absorption principle, we have

$$\begin{aligned}\widehat{\psi}(t,\alpha,y) &= \lim_{\epsilon\to 0+}\frac{1}{2\pi i}\int_{\partial\Omega_\epsilon} e^{-i\alpha tc} i\alpha\Phi(\alpha,y,c)dc\\ &= \frac{1}{2\pi i}\int_{\mathrm{Ran}\,u} e^{-i\alpha tc} i\alpha\Big(\Phi_-(\alpha,y,c)-\Phi_+(\alpha,y,c)\Big)dc\\ &= \frac{1}{2\pi}\int_{\mathrm{Ran}\,u} e^{-i\alpha tc}\widetilde{\Phi}(\alpha,y,c)dc,\end{aligned}$$

where $\widetilde{\Phi}(\alpha,y,c)=\alpha\Big(\Phi_-(\alpha,y,c)-\Phi_+(\alpha,y,c)\Big)$. Then using Plancherel's formula, we infer that

$$\|\widehat{V}(t,\alpha,y)\|^2_{H^1_t L^2_y} \le C\int_{\mathrm{Ran}\,u}\|\widetilde{\Phi}(\alpha,\cdot,c)\|^2_{H^1_y}dc \le C\|\widehat{\omega}_0(\alpha,\cdot)\|^2_{H^1_y}.$$

Linear damping and vorticity depletion

Let $W(\alpha, y, c) = i\alpha(\alpha^2 - \partial_y^2)\Phi(\alpha, y, c)$. We have

$$\widehat{\omega}(t, \alpha, y) = \frac{1}{2\pi i}\int_{\partial\Omega_\epsilon} e^{-i\alpha tc} W(\alpha, y, c)dc.$$

From the proof of the limiting absorption principle, we can deduce that there exists a subsequence $\epsilon_n \to 0+$ and $W_\pm(c) \in L^p_c(\mathrm{Ran}\, u)(1 < p < 4/3)$ so that $W(\alpha, y_c, \cdot \pm i\epsilon_n) \rightharpoonup W_\pm$ weakly in $L^p_c(\mathrm{Ran}\, u)$, where y_c is a critical point. Then using Riemann-Lebesgue lemma, we deduce that as $t \to \infty$,

$$\widehat{\omega}(t, \alpha, y_c) = \frac{1}{2\pi i}\int_{\mathrm{Ran}\, u} e^{-i\alpha tc}\Big(W_-(c) - W_+(c)\Big)dc \to 0.$$

Decay estimates for symmetric flows

To obtain the explicit decay rate of the velocity, we need to know more precise behaviour of the limit function $\Phi_{\pm}(\alpha, y, c)$. To this end, we consider a class of symmetric flows. The key observation is that the solution of (4) can be split into the odd part and even part due to the symmetry of $u(y)$:

$$\begin{cases} \Phi_o'' - \alpha^2 \Phi_o - \dfrac{u''}{u-c} \Phi_o = f_o, \\ \Phi_o(0) = \Phi_o(1) = 0, \end{cases}$$

and

$$\begin{cases} \Phi_e'' - \alpha^2 \Phi_e - \dfrac{u''}{u-c} \Phi_e = f_e, \\ \Phi_e'(0) = \Phi_e(1) = 0, \end{cases}$$

where f_o and f_e are the odd part and even part of f respectively.

Homogeneous Rayleigh equation

These two equations can be dealt as in the monotonic case. The key difference is that $\frac{1}{u(y)-c}$ is more singular for $c = u(0)$ so that the solution $\phi(y, c) = \phi_1(y, c)(u(y) - c)$ of the homogeneous Rayleigh equation

$$\begin{cases} \phi'' - \alpha^2\phi - \frac{u''}{u-c}\phi = 0, \\ \phi(y_c, c) = 0, \quad \phi'(y_c, c) = u'(y_c), \end{cases}$$

only holds the following weighted estimates:

$$|u'(y_c)^\gamma \partial_y^\beta \partial_c^\gamma \phi_1(y, c)| \le C\alpha^{\beta+\gamma}\phi_1(y, c),$$

$$\left|\left(\frac{\partial_y}{u'(y_c)} + \partial_c\right)\phi_1(y, c)\right| \le C\frac{\min\{1, \alpha^2|y - y_c|^2\}\phi_1}{u'(y_c)^2}.$$

From spectral condition to Wronskian condition

We introduce

$$\mathrm{A}(c) = \mathrm{A}_1(c) + u'(y_c)\rho(c)\mathrm{II}_3(c), \quad \mathrm{B}(c) = \pi\rho(c)\frac{u''(y_c)}{u'(y_c)^2},$$

$$\mathrm{A}_2(c) = (u(0)-c)\mathrm{A}(c) + J(c), \quad \mathrm{B}_2(c) = (u(0)-c)\mathrm{B}(c),$$

where

$$\rho(c) = (c-u(0))(u(1)-c), \quad J(c) = \frac{u'(y_c)(u(1)-c)}{\phi_1(0,c)\phi_1'(0,c)},$$

$$\mathrm{A}_1(c) = \rho u'(y_c)\partial_c\Big(p.v.\int_0^1 \frac{dy}{u(y)-c}\Big),$$

$$\mathrm{II}_3(c) = \int_0^1 \frac{1}{(u(y)-c)^2}\Big(\frac{1}{\phi_1(y,c)^2} - 1\Big)dy.$$

Then $c \in D_0$ is an embedding eigenvalue of $\mathcal{R}_\alpha$ if and only if

$$\mathrm{A}(c)^2 + \mathrm{B}(c)^2 = 0 \quad \text{or} \quad \mathrm{A}_2(c)^2 + \mathrm{B}_2(c)^2 = 0.$$

Inhomogeneous Rayleigh equation-odd part

For $c \in \Omega_{\epsilon_0}$, we have

$$\Phi_o(y,c) = \phi(y,c)\int_0^y \frac{1}{\phi(y',c)^2}\int_{y_c}^{y'} f_o\phi(y'',c)dy''dy' + \mu^o(c)\phi(y,c)\int_0^y \frac{1}{\phi(y',c)^2}dy',$$

where

$$\mu^o(c) = \frac{-\int_0^1 \frac{1}{\phi(y',c)^2}\int_{y_c}^{y'} f_o\phi(y'',c)dy''dy'}{\int_0^1 \frac{1}{\phi(y',c)^2}dy'}.$$

Inhomogeneous Rayleigh equation-even part

For $c \in \Omega_{\epsilon_0}$, we have

$$\Phi_e(y,c) = \phi(y,c)\int_0^y \frac{1}{\phi(y',c)^2}\int_{y_c}^{y'} f_e\phi(y'',c)dy''dy' + \mu^e(c)\phi(y,c)\int_0^y \frac{1}{\phi(y',c)^2}dy' + \nu^e(c)\phi(y,c),$$

where μ^e and ν^e are determined by solving

$$\int_0^1 \frac{1}{\phi(y',c)^2}\int_{y_c}^{y'} f_e\phi(y'',c)dy''dy' + \mu^e(c)\int_0^1 \frac{1}{\phi(y',c)^2}dy' + \nu^e(c) = 0,$$

$$\nu^e(c)\phi(0,c)\phi'(0,c) + \int_{y_c}^0 f_e\phi(y'',c)dy'' + \mu^e(c) = 0.$$

We denote

$$\mathrm{C}_e(c) = \rho(c)\frac{\widehat{\omega}_e(y_c)}{u'(y_c)}\pi, \quad \mathrm{D}_e(c) = u'(y_c)\rho(c)\mathrm{II}_1(\widehat{\omega}_e)(c),$$

$$\mathrm{C}_o(c) = \rho(c)\frac{\widehat{\omega}_o(y_c)}{u'(y_c)}\pi, \quad \mathrm{D}_o(c) = u'(y_c)\rho(c)\mathrm{II}_1(\widehat{\omega}_o)(c),$$

$$\mathrm{E}_e(c) = \mathrm{E}(\widehat{\omega}_e)(c) = \int_{y_c}^{0} \widehat{\omega}_e \phi_1(y,c)\,dy,$$

$$\mathrm{II}_1(\varphi)(c) = \mathrm{II}_{1,1}(\varphi)(c) + \mathrm{II}_{1,2}(\varphi)(c),$$

where

$$\mathrm{II}_{1,2}(\varphi)(c) = \int_0^1 \int_{y_c}^{z} \varphi(y)\left(\frac{1}{(u(z)-c)^2}\left(\frac{\phi_1(y,c)}{\phi_1(z,c)^2} - 1\right)\right) dydz.$$

The limit of inhomogeneous Rayleigh equation

The limit of $\Phi_o(y, c \pm i\epsilon)$ when $c \in D_0, \epsilon \to 0+$ takes

$$\Phi_{\pm}^{o}(y,c) = \begin{cases} \phi \int_0^y \frac{1}{\phi(z,c)^2} \int_{y_c}^{z} \phi f_o(y',c)dy'dz \\ \quad +\mu_{\pm}^{o}(c)\phi \int_0^y \frac{1}{\phi(y',c)^2} dy' \quad 0 \le y \le y_c, \\ \phi \int_1^y \frac{1}{\phi(z,c)^2} \int_{y_c}^{z} \phi f_o(y',c)dy'dz \\ \quad +\mu_{\pm}^{o}(c)\phi \int_1^y \frac{1}{\phi(y',c)^2} dy' \quad y_c \le y \le 1, \end{cases}$$

where

$$\mu_{+}^{o}(c) = \frac{1}{\alpha} \frac{-\mathrm{C}_o(c) + i\mathrm{D}_o(c)}{\mathrm{A}(c) - i\mathrm{B}(c)},$$

$$\mu_{-}^{o}(c) = \frac{1}{\alpha} \frac{\mathrm{C}_o(c) + i\mathrm{D}_o(c)}{\mathrm{A}(c) + i\mathrm{B}(c)}.$$

The limit of inhomogeneous Rayleigh equation

The limit of $\Phi_e(y, c \pm i\epsilon)$ when $c \in D_0, \epsilon \to 0+$ takes

$$\Phi^e_\pm(y,c) = \begin{cases} \phi \int_0^y \frac{\int_{y_c}^{y'} \phi f_e(y'',c)dy''}{\phi(y',c)^2} dy' + \mu^e_\pm(c)\phi \int_0^y \frac{1}{\phi(y',c)^2} dy' \\ \quad + \nu^e_\pm(c)\phi(y,c) \quad 0 \le y \le y_c, \\ \phi \int_1^y \frac{\int_{y_c}^{y'} \phi f_e(y'',c)dy''}{\phi(y',c)^2} dy' \\ \quad + \mu^e_\pm(c)\phi \int_1^y \frac{1}{\phi(y',c)^2} dy' \quad y_c \le y \le 1, \end{cases}$$

where

The limit of inhomogeneous Rayleigh equation

$$\mu_+^e(c) = \frac{1}{\alpha}\frac{\phi(0,c)\phi'(0,c)(i\mathrm{D}_e - \mathrm{C}_e)(c) - i\mathrm{E}_e(c)u'(y_c)\rho(c)}{\phi(0,c)\phi'(0,c)(\mathrm{A} - i\mathrm{B})(c) - \rho(c)u'(y_c)},$$

$$\mu_-^e(c) = \frac{1}{\alpha}\frac{\phi(0,c)\phi'(0,c)(i\mathrm{D}_e + \mathrm{C}_e)(c) - i\mathrm{E}_e(c)u'(y_c)\rho(c)}{\phi(0,c)\phi'(0,c)(\mathrm{A} + i\mathrm{B})(c) - \rho(c)u'(y_c)},$$

$$\nu_+^e(c) = -\frac{1}{\alpha}\frac{i\mathrm{D}_e(c) - \mathrm{C}_e(c) - i\mathrm{E}_e(c)(\mathrm{A} - i\mathrm{B})(c)}{\phi(0,c)\phi'(0,c)(\mathrm{A} - i\mathrm{B})(c) - u'(y_c)\rho(c)},$$

$$\nu_-^e(c) = -\frac{1}{\alpha}\frac{i\mathrm{D}_e(c) + \mathrm{C}_e(c) - i\mathrm{E}_e(c)(\mathrm{A} + i\mathrm{B})(c)}{\phi(0,c)\phi'(0,c)(\mathrm{A} + i\mathrm{B})(c) - u'(y_c)\rho(c)}.$$

Note that

$$\phi(0,c)\phi'(0,c)(\mathrm{A} + i\mathrm{B})(c) - u'(y_c)\rho(c) \\ = \phi_1(0,c)\phi_1'(0,c)(u(0) - c)(\mathrm{A}_2 + i\mathrm{B}_2)(c).$$

The formula of the stream function

We introduce

$$
\begin{aligned}
\widetilde{\Phi}(y,c) &= \widetilde{\Phi}_o(y,c) + \widetilde{\Phi}_e(y,c) \\
&= \begin{cases} (\mu_-^o(c) - \mu_+^o(c))\phi(y,c)\displaystyle\int_0^y \frac{1}{\phi(z,c)^2}dz \\ (\mu_-^o(c) - \mu_+^o(c))\phi(y,c)\displaystyle\int_1^y \frac{1}{\phi(z,c)^2}dz \end{cases} \\
&\quad + \begin{cases} (\mu_-^e(c) - \mu_+^e(c))\phi(y,c)\displaystyle\int_0^y \frac{1}{\phi(z,c)^2}dz \\ \qquad + (\nu_-^e(c) - \nu_+^e(c))\phi(y,c) & 0 \le y < y_c, \\ (\mu_-^e(c) - \mu_+^e(c))\phi(y,c)\displaystyle\int_1^y \frac{1}{\phi(z,c)^2}dz & y_c < y \le 1. \end{cases}
\end{aligned}
$$

The formula of the stream function

Based on the above formula and using the limiting absorption principle, we deduce that

$$
\begin{aligned}
\widehat{\psi}(t,\alpha,y) =& \frac{1}{2\pi}\int_{u(0)}^{u(1)} \alpha\widetilde{\Phi}(y,c)e^{-i\alpha ct}dc \\
=& \frac{1}{2\pi}\int_{u(0)}^{u(1)} \alpha\widetilde{\Phi}_o(y,c)e^{-i\alpha ct}dc \\
&+ \frac{1}{2\pi}\int_{u(0)}^{u(1)} \alpha\widetilde{\Phi}_e(y,c)e^{-i\alpha ct}dc \\
=& \widehat{\psi}_o(t,\alpha,y) + \widehat{\psi}_e(t,\alpha,y).
\end{aligned}
$$

Case's formal prediction

Let us consider the monotone flow. In this case, we have

$$\widehat{\psi}(t,\alpha,y) = \frac{1}{2\pi}\int_{u(0)}^{u(1)} \alpha\widetilde{\Phi}(y,c)e^{-i\alpha ct}dc,$$

where $\alpha\widetilde{\Phi}(y,c) = 2\rho(c)\mu(c)\Gamma(y,c)$ with

$$\Gamma(y,c) = \begin{cases} \phi(y,c)\int_0^y \frac{1}{\phi(z,c)^2}dz & 0 \le y < y_c, \\ \phi(y,c)\int_1^y \frac{1}{\phi(z,c)^2}dz & y_c < y \le 1, \end{cases}$$

and $\mu(c)$ is given by

$$\rho(c)\mu(c) = \frac{\mathrm{AC}+\mathrm{BD}}{\mathrm{A}^2+\mathrm{B}^2}.$$

Case's formal prediction is based on the analysis for the singularity of critical layer:

Formally, $\phi(y, c)$ has the behaviour near y_c:

$$\phi(y, c) \sim u'(y_c)(y - y_c) + \frac{u''(y_c)}{2}(y - y_c)^2 + o\big((y - y_c)^2\big),$$

which implies that $\Gamma(y, c)$ has a logarithmic singularity near y_c:

$$\Gamma(y, c) \sim a + b(y - y_c) \log |y - y_c|.$$

Formally, we have

$$\widehat{\psi}(t, \alpha, y) \sim O(t^{-2}) \int_{u(0)}^{u(1)} e^{-i\alpha tc} \alpha \partial_c^2 \widetilde{\Phi}(y, c) dc \sim O(t^{-2}).$$

Decay estimates by dual method

For $f = g'' - \alpha^2 g$ with $g \in H^2(0,1) \cap H_0^1(0,1)$,

$$\int_0^1 \widehat{\psi}_o(t,\alpha,y) f(y) dy = -\int_{u(0)}^{u(1)} K_o(c,\alpha) e^{-i\alpha ct} dc,$$

and for $f = g'' - \alpha^2 g$ with $g \in H^2(0,1)$ and $g'(0) = g(1) = 0$,

$$\int_0^1 \widehat{\psi}_e(t,\alpha,y) f(y) dy = -\int_{u(0)}^{u(1)} K_e(c,\alpha) e^{-i\alpha ct} dc,$$

where

$$K_o(c,\alpha) = \frac{\Lambda_1(\widehat{\omega}_o)(c)\Lambda_2(g)(c)}{(\mathrm{A}(c)^2 + \mathrm{B}(c)^2) u'(y_c)},$$

$$K_e(c,\alpha) = \frac{\Lambda_3(\widehat{\omega}_e)(c)\Lambda_4(g)(c)}{u'(y_c)(\mathrm{A}_2^2 + \mathrm{B}_2^2)(c)}.$$

Dual argument and decay estimates

Now, the problem is reduce to the regularity estimates of K_o and K_e. The main trouble is to cancel the singular factor $\frac{1}{u'(y_c)}$ in the kernel. For this, we have to use many cancellation properties hidden in the kernel.

Proposition 1. Assume that $f = g'' - \alpha^2 g$ with $g \in H^2(0,1) \cap H_0^1(0,1)$ and $\widehat{\omega}_o(\alpha, y) = \frac{1}{2}(\widehat{\omega}_0(\alpha, y) - \widehat{\omega}_0(\alpha, -y)) \in H^2(0,1)$. Then it holds that

$$K_o(u(0), \alpha) = K_o(u(1), \alpha) = 0,$$

and there exists a constant C independent of α so that

$$\|K_o(\alpha, \cdot)\|_{L^1_c} \leq C\|\widehat{\omega}_o(\alpha, \cdot)\|_{L^2_y}\|g\|_{L^2},$$
$$\|(\partial_c K_o)(\alpha, \cdot)\|_{L^1_c} \leq C\|\widehat{\omega}_o(\alpha, \cdot)\|_{H^1_y}\|g\|_{H^1},$$
$$\|(\partial_c^2 K_o)(\alpha, \cdot)\|_{L^1_c} \leq C\alpha^{\frac{1}{2}}\|\widehat{\omega}_o(\alpha, \cdot)\|_{H^2_y}\|f\|_{L^2}.$$

Dual argument and decay estimates

Proposition 2. Assume that $f = g'' - \alpha^2 g$ with $g \in H^2(0,1)$ and $g'(0) = g(1) = 0$, and $\widehat{\omega}_e(\alpha, y) = \frac{1}{2}(\widehat{\omega}_0(\alpha, y) + \widehat{\omega}_0(\alpha, -y)) \in H^2(0,1)$. Then we have

$$K_e(u(0), \alpha) = K_e(u(1), \alpha) = 0,$$

and there exists a constant C independent of α such that

$$\|K_e(\alpha, \cdot)\|_{L^1_c} \leq C\|\widehat{\omega}_e(\alpha, \cdot)\|_{L^2_y}\|g\|_{L^2},$$

$$\|(\partial_c K_e)(\alpha, \cdot)\|_{L^1_c} \leq C\alpha^{\frac{1}{2}}\|\widehat{\omega}_e(\alpha, \cdot)\|_{H^1_y}(\|g'\|_{L^2} + \alpha\|g\|_{L^2}),$$

$$\|(\partial_c^2 K_e)(\alpha, \cdot)\|_{L^1_c} \leq C\alpha^{\frac{3}{2}}\|\widehat{\omega}_e(\alpha, \cdot)\|_{H^2_y}\|f\|_{L^2}.$$

Dual argument and decay estimates

Using Proposition 1,2, we get by integration by parts that

$$\begin{aligned}
\|\widehat{\psi}_o(t,\alpha,\cdot)\|_{L^2_y} &= 2 \sup_{\|f\|_{L^2}=1} \left| \int_0^1 \widehat{\psi}_o(t,\alpha,y) f(y) dy \right| \\
&= 2 \sup_{\|f\|_{L^2}=1} \left| \int_{u(0)}^{u(1)} K_o(c,\alpha) e^{-i\alpha ct} dc \right| \\
&\leq C \frac{1}{\alpha^{\frac{3}{2}} t^2} \|\widehat{\omega}_o(\alpha,\cdot)\|_{H^2_y}, \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
\|\widehat{\psi}_e(t,\alpha,\cdot)\|_{L^2_y} &= 2 \sup_{\|f\|_{L^2}=1} \left| \int_0^1 \widehat{\psi}_e(t,\alpha,y) f(y) dy \right| \\
&= 2 \sup_{\|f\|_{L^2}=1} \left| \int_{u(0)}^{u(1)} K_e(c,\alpha) e^{-i\alpha ct} dc \right| \\
&\leq C \frac{1}{\alpha^{\frac{1}{2}} t^2} \|\widehat{\omega}_o(\alpha,\cdot)\|_{H^2_y}.
\end{aligned}$$

Dual argument and decay estimates

Similar argument gives

$$\alpha^2\|\widehat{\psi}_o(t,\alpha,\cdot)\|_{L^2_y}^2 + \|\partial_y\widehat{\psi}_o(t,\alpha,\cdot)\|_{L^2_y}^2$$

$$= -2\int_0^1 \widehat{\psi}_o(t,\alpha,y)(\overline{\widehat{\psi}_o}'' - \alpha^2\overline{\widehat{\psi}_o})(t,\alpha,y)dy$$

$$\leq \begin{cases} C\|\widehat{\omega}_o(\alpha,\cdot)\|_{L^2_y}\|\widehat{\psi}_o(t,\alpha,\cdot)\|_{L^2_y}, \\ C\dfrac{1}{\alpha t}\|\widehat{\omega}_o(\alpha,\cdot)\|_{H^1_y}\|\widehat{\psi}_o(t,\alpha,\cdot)\|_{H^1_y}, \end{cases}$$

and

$$
\begin{aligned}
&\alpha^2\|\widehat{\psi}_e(t,\alpha,\cdot)\|_{L^2_y}^2 + \|\partial_y\widehat{\psi}_e(t,\alpha,\cdot)\|_{L^2_y}^2 \\
&= -2\int_0^1 \widehat{\psi}_e(t,\alpha,y)(\overline{\widehat{\psi}_e}'' - \alpha^2\overline{\widehat{\psi}_e})(t,\alpha,y)dy \\
&\leq \begin{cases} C\|\widehat{\omega}_e(\alpha,\cdot)\|_{L^2_y}\|\widehat{\psi}_e(t,\alpha,\cdot)\|_{L^2_y}, \\ C\dfrac{1}{\alpha^{\frac{1}{2}}t}\|\widehat{\omega}_e(\alpha,\cdot)\|_{H^1_y}\|\widehat{\psi}_e(t,\alpha,\cdot)\|_{H^1_y}. \end{cases}
\end{aligned}
$$

We consider the Kolmogorov flow $V_K = (-\cos y, 0)$ and the Euler equation on the torus $\left\{(x,y) : x \in \mathbb{T}_{2\pi/\delta}, y \in \mathbb{T}_{2\pi}\right\}$ with $\delta > 1$.

Theorem 5.(Wei-Zhang-Zhao preprint)
Assume that $\int_{\mathbb{T}} \omega_0(x,y)dx = 0$. Then it holds that

1. if $\omega_0(x,y) \in H_x^{-\frac{1}{2}} H_y^1$, then

$$\|V(t)\|_{L^2} \le \frac{C}{\langle t \rangle} \|\omega_0\|_{H_x^{-\frac{1}{2}} H_y^1};$$

2. if $\omega_0(x,y) \in H_x^{\frac{1}{2}} H_y^2$, then

$$\|V^2(t)\|_{L^2} \le \frac{C}{\langle t \rangle^2} \|\omega_0\|_{H_x^{\frac{1}{2}} H_y^2}.$$

Another result is the behavior of the solution at critical points $y = k\pi$ for $k \in \mathbb{Z}$.

Theorem 6.(Wei-Zhang-Zhao preprint)
Assume $\int_{\mathbb{T}} \omega_0(x, y)dx = 0$ and $s > \frac{1}{2}$. Then it holds for all $k \in \mathbb{Z}$ that

1. if $\omega_e(x, y) \in H_x^{1+s} H_y^3$, then for $x \in \mathbb{T}$,
$$|\omega(t, x, k\pi)| \leq \frac{C}{\langle t \rangle} \|\omega_e\|_{H_x^{1+s} H_y^3};$$
2. if $\omega_e(x, y) \in H_x^{1+s} H_y^3$, then for $x \in \mathbb{T}$,
$$|V^2(t, x, k\pi)| \leq \frac{C}{\langle t \rangle^2} \|\omega_e\|_{H_x^{1+s} H_y^3};$$
3. if $\omega_o(x, y) \in H_x^{s-\frac{1}{2}} H_y^3$, then for $x \in \mathbb{T}$,
$$|V^1(t, x, k\pi)| \leq \frac{C}{\langle t \rangle^{\frac{3}{2}}} \|\omega_o\|_{H_x^{s-\frac{1}{2}} H_y^3}.$$

Enhanced dissipation for the Kolmogorov flow

We consider the 2-D Navier-Stokes equations on the torus:

$$\partial_t V - \nu\Delta V + V \cdot \nabla V + \nabla P = 0, \quad \nabla \cdot V = 0.$$

where $0 < \nu \ll 1$.

Numerics and experiment show that the solution of the nearly inviscid 2-D Navier-Stokes equations on the torus will rapidly approach some long-lived quasi-stationary or metastable states such as the Kolmogorov flow $V = (-e^{-\nu t} \cos y, 0)$ called bar state .

The 2D linearized Navier-Stokes equations around the Kolmogorov flow take as follows

$$\partial_t \omega + \mathcal{L}_\nu(t)\omega = 0, \quad \omega|_{t=0} = \omega_0(x, y),$$

where $\mathcal{L}_\nu(t) = -\nu\Delta - e^{-\nu t}(\cos y)\partial_x(1 + \Delta^{-1})$.

Beck and Wayne's conjecture(Proceedings of RSE 2013):

If $\int_{\mathbb{T}} \omega_0(t,x,y)dx = 0$, then for $t \lesssim \frac{1}{\nu}$,

$$\|\omega(t)\|_{L^2} = \mathcal{O}(e^{-c\sqrt{\nu}t}).$$

They verified this conjecture for a toy model using Villan's hypocoercivity method:

$$\partial_t \omega - \nu\Delta\omega - e^{-\nu t}(\cos y)\partial_x \omega = 0.$$

Theorem 7. (Wei-Zhang-Zhao preprint)
Given a $\tau > 0$, there exist $c_1 > 0,\ K > 0$, so that if $\omega_0 \in L^2,\ \int_{\mathbb{T}} \omega_0(x,y)dx = 0$, then for $0 \le t \le \tau/\nu$,

$$\|\omega(t)\|_{L^2} \le Ke^{-c_1\sqrt{\nu}t}\|\omega_0\|_{L^2},$$

$$\|V(t)\|_{\dot{H}^1_x L^2_y} \le \frac{Ke^{-c_1\sqrt{\nu}t}}{\sqrt{1+\nu t^3}}\|\omega_0\|_{L^2}.$$

Thanks for your attention!