An infinite linear hierarchy for the classical Navier-Stokes equation

陈泽乾 (Zeqian Chen)

中国科学院武汉物理与数学研究所
(Wuhan Institute of Physics and Mathematics, CAS)

The International Conference on
Probabilistic Approach to Harmonic Analysis

Wuhan University, Wuhan, 2017-05-21

目录

Navier-Stokes equation

Classical Navier-Stokes equation

The homogeneous, incompressible Navier-Stokes equation in $\mathbb{R}^3$

$$\begin{cases} \partial_t u + (u \cdot \bigtriangledown) u = \triangle u - \bigtriangledown p, \\ \bigtriangledown \cdot u = 0, \end{cases} \tag{1.1}$$

with the initial data $u(0) = u_0$ satisfying $\bigtriangledown \cdot u_0 = 0$.

- $u = u(t, x) \in \mathbb{R}^3$ is the velocity of the fluid at position $x \in \mathbb{R}^3$ and time $t > 0$,
- $p = p(t, x)$ is a scalar field called the pressure of the fluid, and
- $u_0 = u_0(x) \in \mathbb{R}^3, x \in \mathbb{R}^3$, is a given initial velocity vector.

Navier-Stokes equation

Leray's Navier-Stokes equation

$$\begin{cases} \partial_t u = \triangle u - W(u \otimes u), \\ \bigtriangledown \cdot u = 0, \end{cases} \tag{1.2}$$

where $W(u \otimes u) = P \bigtriangledown \cdot (u \otimes u)$ with P being the Leray projection on $[L^2(\mathbb{R}^3)]^3$.

1. J. Leray, Sur le mouvement d'un fluide visqueux emplissant l'espace, *Acta Math.* **63** (1934), 193-248.
2. P. G. Lemarié-Rieusset, *Recent Developments in the Navier-Stokes Problem*, Chapman & Hall/CRC, Boca Raton, 2002.
3. P. G. Lemarié-Rieusset, *The Navier-Stokes Problem in the 21st Century*, CRC Press, Boca Raton, 2016.

Canonical problems in fluid mechanics

Physical problems

- Irreversibility (time).
- Turbulence (space).

Mathematical problems

- Hilbert's 6th problem: How mathematically derives the laws of motion of continua from the atomistic view.
- Global regularity problem: The existence of global smooth solutions to (1.1) for large initial datums.

- Hilbert's 6th problem vs Irreversibility.
- Global regularity problem vs Turbulence.

An infinite linear hierarchy

From the quantum-mechanical point of view, we take

$$u^{(k)}(t,\vec{x}_k) := \otimes_{j=1}^{k} u(t,x_j)$$

for any $k \geq 1$ and for all $\vec{x}_k = (x_1, \ldots, x_k) \in (\mathbb{R}^3)^k$. Then $\mathfrak{U} = (u^{(k)})_{k\geq 1}$ satisfies an infinite hierarchy of linear equations:

Navier-Stokes hierarchy

$$\partial_t u^{(k)}(t,\vec{x}_k) = \sum_{j=1}^{k} \triangle_j u^{(k)}(t,\vec{x}_k) - \sum_{j=1}^{k} W_j u^{(k+1)}(t,\vec{x}_k,x_j) \quad (2.1)$$

for all $k \geq 1$, where $\triangle_j$ and W_j denote respectively the operators $\triangle$ and W acting on $x_j \in \mathbb{R}^3$ for every $j \geq 1$.

Navier-Stokes hierarchy

Duhamel-type expansion

$$
\begin{aligned}
u^{(k)}(t) &= e^{t\triangle^{(k)}} u_0^{(k)} \\
&\quad + \sum_{j=1}^{n} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{j-1}} dt_j e^{(t-t_1)\triangle^{(k)}} W^{(k)} \cdots \\
&\qquad \times e^{(t_{j-1}-t_j)\triangle^{(k+j-1)}} W^{(k+j-1)} e^{t_j \triangle^{(k+j)}} u_0^{(k+j)} \\
&\quad + \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_n} dt_{n+1} e^{(t-t_1)\triangle^{(k)}} W^{(k)} \cdots \\
&\qquad \times e^{(t_n-t_{n+1})\triangle^{(k+n)}} W^{(k+n)} u^{(k+n+1)}(t_{n+1})
\end{aligned}
\tag{2.2}
$$

for every $n \geq 1$, with the convention $t_0 = t$.

$\triangle^{(m)} = \sum_{j=1}^{m} \triangle_j, \quad W^{(m)} = -\sum_{j=1}^{m} W_j.$

Navier-Stokes hierarchy

$k(k+1)\cdots(k+n) \sim n!$ terms in the summation and remainder expressions on the right hand of (2.2).

Tree-graph expansion

$$\begin{aligned} u^{(k)}(t) =& e^{t\triangle^{(k)}} u_0^{(k)} + \sum_{j=1}^{n} \sum_{\mathbb{T}\in\mathfrak{T}_{j,k}} C_{\mathbb{T},t} u_0^{(k+j)} \\ & - \sum_{\mathbb{T}\in\mathfrak{T}_{n+1,k}} \int_0^t ds R_{\mathbb{T},t-s} u^{(k+n+1)}(s) \end{aligned} \tag{2.3}$$

for any $k \geq 1$ and for every $n \geq 1$.

Here, $\mathfrak{T}_{m,k}$ is the set of k-rooted binary trees encoding the collision ways of $k+m$ "particles" with $|\mathfrak{T}_{m,k}| \lesssim C^m$, where C is a constant depending only on k.

Navier-Stokes hierarchy

Solution formula

A solution formula for the Navier-Stokes equation (1.1) with an initial datum $u_0 \in \mathcal{H}^1(\mathbb{R}^3)$:

$$u(t) = e^{t\triangle}u_0 + \sum_{n=1}^{\infty} \sum_{\mathbb{T}\in\mathfrak{T}_{n,1}} C_{\mathbb{T},t} u_0^{\otimes n+1} \tag{2.4}$$

in the sense of distributions for small $t > 0$.

- Every $\mathbb{T} \in \mathfrak{T}_{n,1}$ indicates a kind of processes of two-body interaction of $n+1$ "particles".
- Every $C_{\mathbb{T},t}$ encodes the two-body interaction of "particles" in a concise form.
- This solution formula is a physical expression for the incompressible Navier-Stokes equation.

Interaction operators

Given $k \geq 1$, we define

- $\mathbb{L}^2_{(k)}(\mathbb{R}^3) := \left\{u^{(k)} = \left(u_{i_1,\dots,i_k}\right) : u_{i_1,\dots,i_k} \in L^2(\mathbb{R}^{3k}), 1 \leq i_1, \dots, i_k \leq 3\right\}$.
- $\bigtriangledown_j \cdot u^{(k+1)} := \left(\partial_{x_j^i} u_{i_1,\dots,i_k,i}\right), \quad 1 \leq j \leq k+1.$
- $P_j u^{(k)} := u^{(k)} + \left(R_{x_j^{i_j}} R_{x_j^\ell} u_{i_1,\dots,i_{j-1},\ell,i_{j+1},\dots,i_k}\right), \quad 1 \leq j \leq k.$
- $\left(W_{j,k+1} u^{(k+1)}\right)(\vec{x}_k) := -P_j \bigtriangledown_j \cdot u^{(k+1)}(\vec{x}_k, x_j).$

Interaction operators

$$W^{(k)} := \sum_{j=1}^{k} W_{j,k+1} \tag{3.1}$$

for any $1 \leq j \leq k$.

Interaction operators

$$\mathbb{H}^1_{(k)}(\mathbb{R}^3) := \left\{ u^{(k)} = (u_{i_1,\dots,i_k}) : \ u_{i_1,\dots,i_k} \in [H^1(\mathbb{R}^3)]^{\otimes^k} \right\} \quad (3.2)$$

Proposition

For every $k \geq 1$, $W^{(k)}$ is well defined for all $u^{(k+1)} \in \mathbb{H}^1_{(k+1)}$ in the sense of distributions, such that

$$\left| \langle \phi^{(k)}, W^{(k)} u^{(k+1)} \rangle_{\mathbb{L}^2_{(k)}} \right| \leq C k^{\frac{1}{2}} \|\phi^{(k)}\|_{\mathbb{H}^1_{(k)}} \|u^{(k+1)}\|_{\mathbb{H}^1_{(k+1)}}, \quad (3.3)$$

where $C > 0$ is a universal constant.
Consequently, for any $k \geq 1$ the operator $W^{(k)}$, originally defined on Schwarz functions, can be extended to a bounded operator from $\mathbb{H}^1_{(k+1)}$ into $\mathbb{H}^{-1}_{(k)}$.

Consistent condition

In terms of $W^{(k)}$'s, we can rewrite the Navier-Stokes hierarchy (2.1) as

$$\partial_t u^{(k)}(t) = \triangle^{(k)} u^{(k)}(t) + W^{(k)} u^{(k+1)}(t) \tag{4.1}$$

for all $k \geq 1$.

Consistent condition

A sequence $(u^{(k)})_{k\geq 1} \in \prod_{k\geq 1} \mathbb{H}^1_{(k)}$ is said to be consistent if

$$\langle u^{(k)}, W^+_{j,k+1} u^{(k+1)} \rangle_{\mathbb{L}^2_{(k)}} = 0 \tag{4.2}$$

for every $k \geq 1$ and for all $1 \leq j \leq k$.

Weak solution

Let $0 < T \leq \infty$. A weak solution on $(0, T)$ for the Navier-Stokes hierarchy (4.1) is defined as a sequence of strongly measurable functions $u^{(k)}(t)$ on $(0, T)$ with values in $\mathfrak{H}^1_{(k)}$ for $k \geq 1$, satisfying the following properties:

- For every $k \geq 1$, one has

$$\int_0^T \left[\langle \partial_t \phi + \triangle^{(k)} \phi, u^{(k)} \rangle_{\mathbb{L}^2_{(k)}} + \langle \phi, W^{(k)} u^{(k+1)} \rangle_{\mathbb{L}^2_{(k)}}\right] dt = 0, \tag{4.3}$$

  for any $\phi \in \mathcal{D}_{(k)}((0, T) \times \mathbb{R}^3)$ with the divergence free property that follows

$$\sum_{\ell=1}^{3} \partial_{x_j^\ell} \phi_{i_1, \ldots, i_{j-1}, \ell, i_{j+1}, \ldots, i_k}(t) = 0 \tag{4.4}$$

  for all $0 < t < T$ and for every $1 \leq i_1, \ldots, i_k \leq 3$.

Weak solution-continued

- For any $k \geq 1$ the divergence free conditions

$$\sum_{\ell=1}^{3} \partial_{x_j^\ell} u^{(k)}_{i_1,\ldots,i_{j-1},\ell,i_{j+1},\ldots,i_k}(t) = 0 \tag{4.5}$$

  hold true for every $t \in (0, T)$ and all $1 \leq j \leq k$, where $1 \leq i_1, \ldots, i_k \leq 3$.
- The sequences $(u^{(k)}(t))_{k \geq 1}$ are consistent for every $t \in (0, T)$.

As for $T = \infty$, the solution is called a global weak solution.

Weak solution for Cauchy problem

Let $0 < T \leq \infty$. Suppose $\mathfrak{U}_0 = (u_0^{(k)})_{k\geq 1} \in \prod_{k\geq 1} \mathfrak{L}^2_{(k)}(\mathbb{R}^3)$ such that for every $k \geq 1$,

$$\sum_{\ell=1}^{3} \partial_{x_j^\ell} (u_0^{(k)})_{i_1,\ldots,i_{j-1},\ell,i_{j+1},\ldots,i_k} = 0 \tag{4.6}$$

in the sense of distributions for all $j = 1, \ldots, k$, and for any $1 \leq i_1, \ldots, i_k \leq 3$. A weak solution on $[0, T)$ for the Cauchy problem of the Navier-Stokes hierarchy (4.1) with the initial data $\mathfrak{U}_0$ that is

$$u^{(k)}(0) = u_0^{(k)} \tag{4.7}$$

for any $k \geq 1$, is by definition a weak solution $(u^{(k)}(t))_{k\geq 1}$ on $(0, T)$ to the Navier-Stokes hierarchy (4.1) such that for every $k \geq 1$, $u^{(k)} \in L^\infty\big((0, T); \mathbb{L}^2_{(k)}(\mathbb{R}^3)\big)$ and $\lim_{t\to 0} u^{(k)}(t) = u_0^{(k)}$ in the weak topology of $\mathbb{L}^2_{(k)}(\mathbb{R}^3)$.

Integral Navier-Stokes hierarchy

The integral Navier-Stokes hierarchy that follows

$$u^{(k)}(t) = \mathcal{T}^{(k)}(t)u_0^{(k)} + \int_0^t ds \mathcal{T}^{(k)}(t-s)W^{(k)}u^{(k+1)}(s) \qquad (4.8)$$

for all $k \geq 1$, where the free evolution operator $\mathcal{T}^{(k)}(t)$ is defined on $\mathbb{L}^2_{(k)}$ for every $t \geq 0$ by

$$\mathcal{T}^{(k)}(t)u^{(k)} := \left(e^{t\triangle^{(k)}} u_{i_1,\ldots,i_k}\right) \qquad (4.9)$$

for every $u^{(k)} = \left(u_{i_1,\ldots,i_k}\right) \in \mathbb{L}^2_{(k)}$.

Mild solution for integral Navier-Stokes hierarchy

Let $0 < T \leq \infty$. Suppose $\mathfrak{U}_0 = (u_0^{(k)})_{k\geq 1} \in \prod_{k\geq 1} \mathfrak{L}^2_{(k)}(\mathbb{R}^3)$ satisfies the divergence-free condition (4.6) for all $k \geq 1$. A mild solution on $[0, T)$ to the integral Navier-Stokes hierarchy (4.8) with the prescribed initial condition $\mathfrak{U}_0$ is defined as a sequence of strongly measurable functions $u^{(k)}(t)$ on $(0, T)$ with values in $\mathfrak{H}^1_{(k)}(\mathbb{R}^3)$ for $k \geq 1$, satisfying the following conditions:

1) The integral equation (4.8) holds for all $t \in (0, T)$ in the sense of distributions.
2) The sequences $(u^{(k)}(t))_{k\geq 1}$ are consistent for every $t \in (0, T)$.
3) For every $k \geq 1$, $u^{(k)} \in L^\infty\big((0, T); \mathbb{L}^2_{(k)}(\mathbb{R}^3)\big)$ and

$$\lim_{t\to 0} u^{(k)}(t) = u_0^{(k)}$$

in the weak topology of $\mathbb{L}^2_{(k)}(\mathbb{R}^3)$.

Equivalence

Weak solution vs mild solution

Let $u_0^{(k)} \in \mathfrak{L}^2_{(k)}(\mathbb{R}^3)$ satisfying (4.6) for all $k \geq 1$. A sequence of strongly measurable functions $u^{(k)}(t)$ on $(0, T)$ with values in $\mathfrak{H}^1_{(k)}(\mathbb{R}^3)$ for $k \geq 1$ is a weak solution to the Cauchy problem of the Navier-Stokes hierarchy (4.1) with the initial data $(u^{(k)}(0))_{k\geq 1} = (u_0^{(k)})_{k\geq 1}$, if and only if it is a mild solution to the integral Navier-Stokes hierarchy (4.8) with the prescribed initial condition $(u_0^{(k)})_{k\geq 1}$.

Uniqueness

Uniqueness of mild solution

Let $T > 0$. Assume that $\mathfrak{U}_0 = (u_0^{(k)})_{k\geq 1} \in \prod_{k\geq 1} \mathfrak{H}^1_{(k)}$ is consistent and for every $k \geq 1$, $u_0^{(k)}$ satisfies the divergence-free condition (4.6) and

$$\|u_0^{(k)}\|_{\mathbb{H}^1_{(k)}} \leq C^k \tag{5.1}$$

where $C > 0$ is a constant independent of k. Then the integral Navier-Stokes hierarchy (4.8) has at most one mild solution $\mathfrak{U}(t) = (u^{(k)}(t))_{k\geq 1}$ in $[0, T)$ with $\mathfrak{U}(0) = \mathfrak{U}_0$ such that for every $k \geq 1$, $u^{(k)} \in L^\infty([0,T), \mathfrak{H}^1_{(k)})$ and satisfies the bound

$$\|u^{(k)}\|_{L^\infty([0,T),\mathbb{H}^1_{(k)})} \leq C^k. \tag{5.2}$$

F-space

We define

$$\mathfrak{H}^1_{(\infty)} = \left\{ (u^{(k)})_{k\geq 1} \in \prod_{k\geq 1} \mathfrak{H}^1_{(k)} : \exists \lambda > 0, \sum_{k\geq 1} \frac{1}{\lambda^k} \|u^{(k)}\|_{\mathbb{H}^1_{(k)}} < \infty \right\}$$

equipped with

$$\|(u^{(k)})_{k\geq 1}\|_{\mathfrak{H}^1_{(\infty)}} = \inf \left\{ \lambda > 0 : \sum_{k\geq 1} \frac{1}{\lambda^k} \|u^{(k)}\|_{\mathbb{H}^1_{(k)}} \leq 1 \right\}.$$

Note that $\|\cdot\|_{\mathfrak{H}^1_{(\infty)}}$ is not actually a norm but a (F)-norm in $\mathfrak{H}^1_{(\infty)}$. Thus, $\mathfrak{H}^1_{(\infty)}$ is a F-space.

For any $u \in \mathbb{H}^1(\mathbb{R}^3)$, it is easy to check that $(u^{\otimes^k})_{k\geq 1} \in \mathfrak{H}^1_{(\infty)}$ and

$$\|(u^{\otimes^k})_{k\geq 1}\|_{\mathfrak{H}^1_{(\infty)}} = 2\|u\|_{\mathbb{H}^1}.$$

Namely, $\|\cdot\|_{\mathfrak{H}^1_{(\infty)}}$ is compatible with the Sobolev norm of $\mathbb{H}^1$ for factorized hierarchies $\mathfrak{U} = (u^{(k)})_{k\geq 1}$.

Navier-Stokes hierarchy vs equation

Equivalence between Navier-Stokes equation and hierarchy

Let $u_0 \in \mathbb{H}^1(\mathbb{R}^3)$ such that $\nabla \cdot u_0 = 0$. Let $u(t)$ be the unique (mild) solution in $C([0,T^*), \mathbb{H}^1(\mathbb{R}^3))$ for the Navier-Stokes equation (1.2) with the initial datum $u(0) = u_0$, where T^* is the maximal life-time of $u(t)$. Let $u^{(k)}(t) = u(t)^{\otimes^k}$ for every $k \geq 1$. Then $\mathfrak{U}(t) = (u^{(k)}(t))_{k\geq 1}$ is a unique weak solution in $\mathfrak{H}^1_{(\infty)}$ for the Cauchy problem of the Navier-Stokes hierarchy (4.1) on $[0, T^*)$ with the initial datum $\mathfrak{U}(0) = (u_0^{\otimes^k})_{k\geq 1}$.

This shows that the initial problem for the Navier-Stokes hierarchy (4.1) in $\mathfrak{H}^1_{(\infty)}$ with a factorized divergence-free initial datum is equivalent to the Cauchy problem of the Navier-Stokes equation (1.2) in $\mathcal{H}^1$.

Binary tree

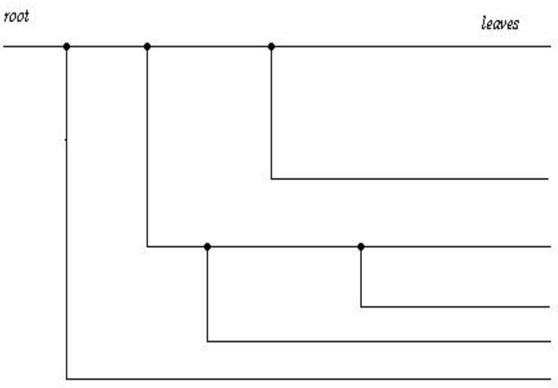

Figure: Example of a rooted, marked binary tree with $n = 5$ vertices.

Binary tree-Forest

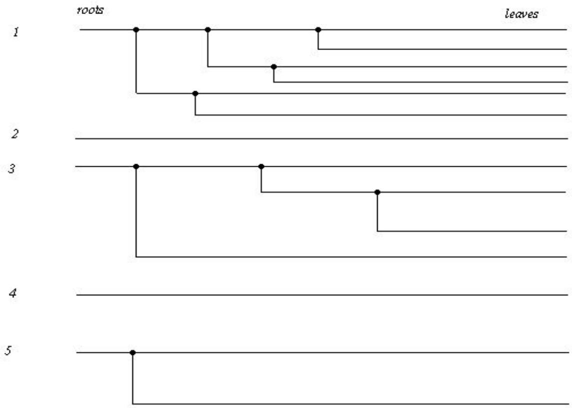

Figure: Example of a forest with $n = 9$ and $k = 5$.

Tree-graphic representation for collision operator

The collision operator $C_{\mathbb{T},t,\Upsilon} : \mathbb{L}^2_{(n+k)} \mapsto \mathbb{L}^2_{(k)}$ is defined by

$$\left(C_{\mathbb{T},t,\Upsilon} u^{(n+k)}\right)(\vec{p}_k) = \int d\vec{r}_{n+k} \left(G_{\mathbb{T},t,\Upsilon} u^{(n+k)}\right)(\vec{p}_k; \vec{r}_{k+n}) \qquad (6.1)$$

through its kernel

$$\begin{aligned}
&\left(G_{\mathbb{T},t,\Upsilon} u^{(n+k)}\right)(\vec{p}_k; \vec{r}_{k+n}) \\
=&\frac{1}{(k+n)!} \sum_{\pi_2 \in \Pi_{k+n}} \prod_{e \in R_1(\mathbb{T}) = L_1(\mathbb{T})} e^{-tp^2_{\pi_1(e)}} \delta(p_{\pi_1(e)} - r_{\pi_2(e)}) \\
&\times \int \prod_{e \in E_2(\mathbb{T})} dq_e d\tau^e K^{\pi_2}_{\mathbb{T}} u^{(n+k)}(\vec{r}_{n+k}) \prod_{e \in R_2(\mathbb{T})} e^{-t(\gamma^e + i\tau^e)} \delta(q_e - p_{\pi_1(e)}) \\
&\times \prod_{v \in V(\mathbb{T})} \delta\left(\tau^{e^a_v} - \tau^{e^b_v} - \tau^{e^c_v}\right) \delta\left(q_{e^a_v} - q_{e^b_v} - q_{e^c_v}\right) \\
&\times \prod_{e \in L_2(\mathbb{T})} \delta(q_e - r_{\pi_2(e)}) \prod_{e \in E_2(\mathbb{T})} \frac{1}{\gamma^e - q^2_e + i\tau^e}.
\end{aligned}$$

Tree-graphic expansion

Proposition

Let $k \geq 1$ and $n \geq 1$. For any given $u^{(k+n)} \in \mathbb{L}^2_{(k+n)}$, we have

$$\sum_{\mathbb{T} \in \mathfrak{T}_{n,k}} C_{\mathbb{T},t} u^{(k+n)} = \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \mathcal{T}^{(k)}(t-t_1) W^{(k)} \cdots \times \mathcal{T}^{(k+n-1)}(t_{n-1}-t_n) W^{(k+n-1)} \mathcal{T}^{(k+n)}(t_n) u^{(k+n)} \tag{6.3}$$

for all $t \geq 0$.

Tree-graphic expansion-continued

Proposition

Let $k \geq 1$ and $n \geq 1$. For any given $T > 0$, if $u^{(k+n)} \in L^2([0,T], \mathfrak{H}^1_{(k+n)})$ then

$$
\begin{aligned}
&- \sum_{\mathbb{T} \in \mathfrak{T}_{n,k}} \int_0^t ds R_{\mathbb{T},t-s} u^{(k+n)}(s) \\
&= \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \mathcal{T}^{(k)}(t-t_1) W^{(k)} \cdots \\
&\quad \times \mathcal{T}^{(k+n-1)}(t_{n-1}-t_n) W^{(k+n-1)} u^{(k+n)}(t_n)
\end{aligned}
\tag{6.4}
$$

for all $t \in [0,T]$.

Space-time estimates for collision operator

Proposition

Fix $\alpha > \frac{1}{2}$ and $k \geq 1$. Let $M > 0$. Then there exists a constant $C > 0$ depending only on α, k such that for any $v^{(k)} \in \mathcal{S}_{(k)}(\mathbb{R}^3)$ satisfying

$$\sup_{1\leq i_1,\ldots,i_k\leq 3}\ \sup_{p_1,\ldots,p_k\in\mathbb{R}^3} \langle p_1\rangle^3 \cdots \langle p_k\rangle^3 |v^{(k)}_{i_1,\ldots,i_k}(\vec{p}_k)| \leq M, \tag{7.1}$$

for any $n \geq 0$, and for all $\mathbb{T} \in \mathfrak{T}_{n,k}$, we have

$$\left|\left\langle v^{(k)}, C_{\mathbb{T},t} u^{(n+k)} \right\rangle_{\mathbb{L}^2_{(k)}}\right| \leq M C^n \left\| u^{(n+k)} \right\|_{\mathbb{H}^\alpha_{(n+k)}} \tag{7.2}$$

for all $u^{(n+k)} \in \mathfrak{H}^\alpha_{(n+k)}$ and any $0 < t \leq 1$.

Space-time estimates for error operator

Proposition

Fix $k \geq 1$. Let $M > 0$. Then there exists a constant $C > 0$ depending only on k such that for any $v^{(k)} \in \mathcal{S}_{(k)}(\mathbb{R}^3)$ satisfying (7.1) with the bound M, for any $n \geq 0$ and any $\mathbb{T} \in \mathfrak{T}_{n,k}$, we have

$$\left|\left\langle v^{(k)}, R_{\mathbb{T},t} u^{(n+k)} \right\rangle_{\mathbb{L}^2_{(k)}}\right| \leq M C^n t^{\frac{n}{2}-1} \left\| u^{(n+k)} \right\|_{\mathbb{H}^1_{(n+k)}} \tag{7.3}$$

for all $u^{(n+k)} \in \mathfrak{H}^1_{(n+k)}$ and any $0 < t \leq 1$.

A solution formula

Theorem

Let $u_0 \in \mathbb{H}^1(\mathbb{R}^3)$ with $\bigtriangledown \cdot u_0 = 0$. Let u be the unique weak solution in $C([0, T^*), \mathbb{H}^1(\mathbb{R}^3))$ for the initial problem of the Navier-Stokes equation

$$\begin{cases} \partial_t u = \triangle u - W(u \otimes u), \\ \bigtriangledown \cdot u = 0, \end{cases} \tag{7.4}$$

with the initial datum $u(0) = u_0$, where T^* is the maximal life-time of $u(t)$. Then there exits $0 < t^* < T^*$ such that

$$u(t) = e^{t\triangle} u_0 + \sum_{n=1}^{\infty} \sum_{\mathbb{T} \in \mathfrak{T}_{n,1}} C_{\mathbb{T},t} u_0^{\otimes^{n+1}} \tag{7.5}$$

in the sense of distributions for every $0 < t < t^*$.

Concluding remarks

In conclusion, we have

- Wave/Field perspective about the classical Navier-Stokes equation.
- Quantum version of Hilbert's 6th problem: From N-partite Schrödinger equations to the classical Euler and Navier-Stokes equations.
- Does the first principle imply the global regularity?

Thank you for your attention!