Abstract Pseudo-Differential Calculus on quantum Euclidean Spaces

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Based on discussions with Hadrian Quan, Marius Junge, and Edward McDonald.

ΨDO operators on R^d

- Derivative operators

$$D^\alpha e^{2\pi i x\cdot\xi} = \xi^\alpha e^{2\pi i x\cdot\xi}\ ,\ D_j = -i\partial_j\ .$$

- Pseudo-Differential operators (ΨDO)

$$T_a(e^{2\pi i x\cdot\xi}) = a(x,\xi)e^{2\pi i x\cdot\xi}\ ,\ (T_a f)(x) = \int_{\mathbb{R}^d} a(x,\xi)\hat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi\ .$$

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- Symbol $a(x,\xi) \xrightarrow{Op}$ Operator T_a
  - $a = a_1(x)$ gives a *multiplication operator* T_{a_1}.
  - $a = a_2(\xi)$ gives a *multiplier operator* T_{a_2}.
  - $a = a_1(x)a_2(\xi)$ gives $T_{a_1}T_{a_2}$, where differentiation precedes multiplication.
- The Op map is a functional calculus of the Heisenberg relation

$$[D_j, x_j] = -iI \ .$$

Pseudo-differential symbol calculus

$$a(x,\xi) \xrightarrow{Op} (T_a f)(x) = \int_{\mathbb{R}^d} a(x,\xi) f(\xi) e^{2\pi i x\cdot\xi} d\xi \ , \ f \in \mathcal{S}(\mathbb{R}^d)$$

- **Symbol class:** $a \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$ is a symbol of order m, or $a \in S^m$, if for any multi-indices α, β, there exists constants $C_{\alpha,\beta}$ s.t.

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|} \ .$$

- **Symbol calculus:** Given $a \in S^n, b \in S^m$, the operator $T_a \circ T_b = T_c$ where c is a symbol of order $n+m$ s.t. $c \sim \sum_\alpha \frac{(2\pi i)^{-|\alpha|}}{|\alpha|!} D_\xi^\alpha(a) D_x^\alpha(b)$ in the sense that

$$c - \sum_{|\alpha|\le N} \frac{(2\pi i)^\alpha}{|\alpha|!} D_\xi^\alpha(a) D_x^\alpha(b) \in S^{n+m-N} \ .$$

- L_2-**boundedness:** $a \in S^0$ implies $T_a \in B(L_2(\mathbb{R}^d))$.

Motivation

- Kernel $\longleftrightarrow$ symbol $\longleftrightarrow$ operator (see Professor Parcet's talk)
- Form Atiyah-Singer Index theorem: [Kohn-Nirenberg,65], [Hörmander,65] + others
- Many applications in PDE: [Taylor,81]+more
- The local index formula by [Connes-Moscovici,95]: Abstract ΨDO calculus in a spectral triple (A, H, D).

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  - D is an (unbounded) self-adjoint operator on a Hilbert space H. Denote that $\delta(a) = [|D|, a]$. A subalgebra $A \subset B(H)$ satisfies the smooth condition: $\delta^n(a)$ bounded for any n and $a \in A, [D, A]$. For $s \in \mathbb{R}$, put $H_s = dom(|D|^s)$ as analog of Sobolev spaces
  - A operator $a \in Op^m$ if $a : H_s \to H_{s-m}$ continuous for all $s \in \mathbb{R}$ $\iff |D|^s a |D|^{-s-m}$ is bounded for all s.

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Question: Abstract symbol calculus?

"Second Quantization"

Classic	$a(x,\xi)$, $C^\infty(\mathbb{R}^d, \mathbb{R}^d)$ x_j, ξ_j commutative	$\longrightarrow$	$T_a \in B(L^2(\mathbb{R}^d))$ $[\xi_j, x_k] = -i\delta_{jk}$
	$\lambda_\theta \otimes id$ $\downarrow$		
[GJP17]	$a(\xi)$, $C^\infty(\mathbb{R}^d, \mathbb{R}_\theta)$ $[x_j, x_k] = -i\theta_{jk}$, ξ_j commutative	$\longrightarrow$	$Op(a) \in B(L^2(\mathbb{R}_\theta))$ $[\xi_j, x_k] = -i\delta_{jk}$, $[x_j, x_k] = -i\theta_{jk}$ ξ_j commutative
	$id \otimes \lambda_{\theta'}$ $\downarrow$		
Our setting	$a_1 \otimes a_2 \in \mathbb{R}_\theta \overline{\otimes} \mathbb{R}_{\theta'}$ $[x_j, x_k] = -i\theta_{jk}$, $[\xi_j, \xi_k] = -i\theta'_{jk}$.	$\longrightarrow$	$Op(a_1 \otimes a_2) = a_1 a_2 \in \mathbb{R}_\Theta$ $[\xi_j, x_k] = -i\delta_{jk}$, $[x_j, x_k] = -i\theta_{jk}$, $[\xi_j, \xi_k] = -i\theta'_{jk}$.

Quantum Euclidean spaces/Moyal Planes

Let θ be a skew-symmetric real $d \times d$ matrix. $\mathbb{R}_\theta$ is the universal vNa generated by $[x_j, x_k] = -i\theta_{jk}$ (CCR relations), or $\lambda_\theta(\xi)\lambda_\theta(\eta) = e^{\frac{i}{2}\xi\cdot\theta\eta}\lambda_\theta(\xi+\eta)$ (projective representation).

- Type I algebra: if $rank(\theta) = 2n$, $\mathbb{R}_\theta \cong B(L_2(\mathbb{R}^n))\overline{\otimes}L_\infty(\mathbb{R}^{d-2n})$.
- Schwartz class: $S(\mathbb{R}_\theta)$ is given by the Weyl quantization
$$\lambda_\theta(f) = \int \hat{f}(\xi)\lambda_\theta(\xi)d\xi \ , f \in \mathcal{S}(\mathbb{R}^d) \ .$$
- Canonical trace: $\tau(\lambda_\theta(f)) = \frac{1}{(2\pi)^d}\hat{f}(0) = \int f(x)dx \ , f \in \mathcal{S}(\mathbb{R}^d)$

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- Canonical trace: $\tau(\lambda_\theta(f)) = \frac{1}{(2\pi)^d}\hat{f}(0) = \int f(x)dx \ , f \in \mathcal{S}(\mathbb{R}^d)$
- Multiplier algebra:
$\mathcal{M}(\mathbb{R}_\theta) = \{a \in \lambda_\theta(\mathcal{S}'(\mathbb{R}^d)) |\ a\mathcal{S}(\mathbb{R}_\theta) \subset \mathcal{S}(\mathbb{R}_\theta), \mathcal{S}(\mathbb{R}_\theta)a \subset \mathcal{S}(\mathbb{R}_\theta)\}$.
- Transference group: $\alpha : \mathbb{R}^d \to Aut(\mathbb{R}_\theta)$,
$$\alpha_x(\lambda_\theta(f)) = \lambda_\theta(\alpha_x(f)), \alpha_x(f)(\cdot) = f(\cdot + x) \ .$$
- Differentiation: $D_j(\lambda_\theta(f)) = \lambda_\theta(D_j(f)) = \lim_{x_j\to 0} \frac{\alpha_{x_j}(\lambda_\theta(f)) - \lambda_\theta(f)}{x_j}$.
- NC variables: $x_j = \lambda_\theta(x_j) \ , x_j\lambda_\theta(f) = \lambda_\theta(x_j f) + \frac{i}{2}\sum_k \theta_{jk}\lambda_\theta(D_k f)$
$[x_j, \lambda_\theta(f)] = i\sum_k \theta_{jk}\lambda_\theta(D_k f)$.

Noncommutative Symbols

Recall our "Op" map

$$Op : \mathbb{R}_{\theta,\theta'} \to \mathbb{R}_\Theta \ , \ Op(a_1 \otimes a_2) = a_1 a_2$$

where $\mathbb{R}_{\theta,\theta'} = \mathbb{R}_\theta \overline{\otimes} \mathbb{R}'_\theta$ generated by

$$[x_j, x_k] = -i\theta_{jk} \ , [\xi_j, \xi_k] = -i\theta'_{jk} \ , [\xi_j, x_k] = 0$$

and $\mathbb{R}_\Theta$ is a $2d$-dimensional NC Euclidean space replacing $[\xi_j, x_k] = 0$ by $[\xi_j, x_k] = -i\delta_{jk}$. Take $\langle \xi \rangle = (1 + \sum_j \xi_j^2)^{1/2}$.

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Definition

$a \in \mathcal{M}(\mathbb{R}_{\theta,\theta'})$ is a symbol of order m, denoted by $a \in S^m$ if for any multi-indices α and β, there exists some constant $C_{\alpha\beta}$ s.t.

$$\| D_x^\alpha D_\xi^\beta (a) \langle \xi \rangle^{|\beta| - m} \|_{\mathbb{R}_{\theta,\theta'}} \leq C_{\alpha\beta} \ .$$

Asymptotic order in R_θ

$\langle\xi\rangle = (1+\sum_j \xi_j^2)^{1/2}$. For $s\in\mathbb{R}$, put $H_s = Dom(\langle\xi\rangle^s)$ with the inner product $\langle v,u\rangle_{H_s} = \tau(v^*\langle\xi\rangle^{2s}u)$. $H_0 = L_2(\mathbb{R}_{\theta'})$, $H^\infty = \cap_s H_s$ is a dense subspace of $L_2(\mathbb{R}_{\theta'})$.

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Let $m \in \mathbb{R}$ and a be closable operator on H^∞. TFAE

- $\langle\xi\rangle^s a \langle\xi\rangle^{-s-m}$ is bounded for any $s \in \mathbb{R}$;
- a extends to bounded operator from H_s to H_{s-m} for any $s \in \mathbb{R}$.

and we denote by O^m ("order m") for all such operators.

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Let $m\in\mathbb{R}$ and a be closable operator on H^∞. TFAE

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Proposition

For all multi-indices α and $s\in\mathbb{R}$,

$$\xi^\alpha \in O^{|\alpha|}\ ,\ [\xi^\alpha,\langle\xi\rangle^s]\in O^{s+|\alpha|-2}\ ,\ D^\alpha(\langle\xi\rangle^s)\in O^{s-|\alpha|}\ .$$

Noncommutative Symbols

Proposition

$a \in S^m$ if and only if for any α, β and $s, t \in \mathbb{R}$,

$$\langle x\rangle^s \langle \xi\rangle^t D_x^\alpha D_\xi^\beta (a) \langle x\rangle^{-s} \langle \xi\rangle^{-m-t+|\beta|}$$

is bounded. (Or denoted by $D_x^\alpha D_\xi^\beta (a) \in O^{0,m-|\beta|}$).

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Corollary

i) $\xi^\alpha \in S^{|\alpha|}$, $\langle\xi\rangle^r \in S^r$;

ii) $a \in S^m$ *if and only if* $a^* \in S^m$;

iii) $a \in S^m, b \in S^n$, *then* $ab \in S^{m+n}$.

Co-multiplication I

We borrow the idea of co-multiplication from quantum groups. Let

$$(\lambda_0(\eta)f)(x) = e^{2\pi i x\cdot\eta} f(x), f \in L_2(\mathbb{R}^d)$$

be the multiplication unitary. The map $\sigma : \mathbb{R}_\theta \to L_\infty(\mathbb{R}^d, \mathbb{R}_\theta)$

$$\sigma_\theta(\lambda_\theta(\eta)) = \lambda_0(\eta) \otimes \lambda_\theta(\eta) \ , \sigma_\theta(x_j) = y_j \otimes 1 + 1 \otimes x_j,$$

$$\sigma_\theta(\lambda_\theta(f)) = \int_{\mathbb{R}^d} \hat{f}(\eta)\lambda_0(\eta) \otimes \lambda_\theta(\eta) d\eta,$$

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$$\sigma_\theta(\lambda_\theta(f)) = \int_{\mathbb{R}^d} \hat{f}(\eta)\lambda_0(\eta) \otimes \lambda_\theta(\eta) d\eta,$$

- an injective $*$-homomorphism from $\mathbb{R}_\theta$ to $L_\infty(\mathbb{R}^d, \mathbb{R}_\theta)$
- $\sigma_\theta(a)(y) = \alpha_y(a)$ and

$$\sigma_\theta(\lambda_\theta(\mathcal{D}_j f)) = [1 \otimes \mathcal{D}_{x_j}, \sigma(\lambda_\theta(f))] = [D_{y_j} \otimes 1, \sigma(\lambda_\theta(f))] \ ,$$

Co-multiplication II

Let

$$(u(y)f)(x) = f(x+y)\ , (\lambda_0(\eta)f)(x) = e^{2\pi i x\cdot\eta} f(x)$$

be the shifting unitary and respectively multiplication unitary. Then the map $\sigma_\Theta : \mathbb{R}_\Theta \to B(L_2(\mathbb{R}^d))\overline{\otimes}\mathbb{R}_\theta\overline{\otimes}\mathbb{R}_{\theta'}$,

$$\sigma_\Theta(\lambda_\Theta(\eta, y)) = \lambda_0(\eta)u(y) \otimes \lambda_\theta(\eta) \otimes \lambda_{\theta'}(y),$$

$$\sigma_\Theta(\lambda_\Theta(F)) = \int_{\mathbb{R}^d} \hat{F}(\eta, y)\big(\lambda_0(\eta)u(y) \otimes \lambda_\theta(\eta) \otimes \lambda_{\theta'}(y)\big)d\eta dy$$

is an injective $*$-homomorphism.

Calderón-Vaillancourt Theorem

The following diagram commutes

$$
\begin{array}{ccccc}
\mathbb{R}_\theta \overline{\otimes} \mathbb{R}_{\theta'} & \xrightarrow{\quad Op \quad} & \mathbb{R}_\Theta & & \\
\downarrow{\scriptstyle \sigma_\theta \otimes \sigma_{\theta'}} & & & \searrow^{\sigma_\Theta} & \\
L_\infty(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}_\theta \bar{\otimes} \mathbb{R}_{\theta'}) & \xrightarrow{Op \otimes id} & \mathcal{L}(L_2^c(\mathbb{R}^d) \otimes_{min} \mathbb{R}_\theta \bar{\otimes} \mathbb{R}_{\theta'}) & \longleftrightarrow & B(L_2(\mathbb{R}^d)) \overline{\otimes} \mathbb{R}_\theta \overline{\otimes} \mathbb{R}_{\theta'}
\end{array}
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- $\sigma_\theta \otimes \sigma_\theta'$ send 0-order symbols to operator-valued 0-order symbols.
- The operator-valued Calderón-Vaillancourt Theorem for Hilbert module maps was proved by [Merklen05] .
- The $\mathbb{R}_\theta \otimes \mathbb{R}_{\theta'}$ module maps $\mathcal{L}(L_2(R^d) \otimes_{min} \mathbb{R}_\theta \bar{\otimes} \mathbb{R}_{\theta'}) \cong B(L_2(\mathbb{R}^n)) \overline{\otimes} \mathbb{R}_\theta \overline{\otimes} \mathbb{R}_{\theta'}$.

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Theorem

The operator map $Op(a_1 \otimes a_2) = a_1 a_2$ extends to a continuous map from the Fréchet space $S^0 \subset \mathbb{R}_\theta \otimes \mathbb{R}_{\theta'}$ to $\mathbb{R}_\Theta$. Moreover,

$$\| Op(a) \| \lesssim \sup\{\| D_x^\alpha D_\xi^\beta (a) \| \ , \ \alpha, \beta \le (1, 1, \cdots, 1)\} \ .$$

Composition identity

Theorem

Let $a \in S^m$ *and* $b \in S^n$*. Then the operator* $Op(a)Op(b) = Op(c)$ *is a pseudo-differential operator of order* $m+n$ *and its symbol*

$$c \sim \sum_{\beta} \frac{(2\pi i)^{-|\beta|}}{|\beta|!} D_\xi^\beta(a) D_x^\beta(b)$$

in the sense that for any N

$$c - \sum_{|\beta| \leq N} \frac{(2\pi i)^{-|\beta|}}{|\beta|!} D_\xi^\beta(a) D_x^\beta(b) \in S^{n+m-N} .$$

Proof. We adapt a classic argument by Stein to the noncommutative setting.

Sketch of proof

- $c(x,\xi) = \int a(x,\eta)b(y,\xi)e^{2\pi i(\eta-\xi)\cdot(x-y)}d\eta dy$ reformulated to

$$c = \int \alpha_\eta(a)\alpha_y(b)e^{-2\pi i\eta\cdot y}d\eta dy \ = \lim_{\epsilon\to 0}\int \alpha_\eta(a)b_\epsilon(y)e^{-2\pi i\eta\cdot y}d\eta dy,$$

where $b_\epsilon(y)$ compactly supported $\to \alpha_y(b)$.

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- Write $c = c_1 + c_2$ where $c_1 = \int \alpha_\eta(a)\hat{b}_1(\eta)d\eta$ with $b_1(y)$ compactly supported.
- Taylor expansion:

$$\alpha_\eta(a) = \sum_{|\alpha|\le N}\frac{(D_\xi^\beta a)\eta^\alpha}{\beta!} + \sum_{|\beta|=N+1}\frac{1}{N!}\int_0^1 \alpha_{t\eta}(D_\xi^\beta a)(1-t)^N dt\ .$$

- $$c_1 = \int \sum_{|\beta|\le N}\frac{D_\xi^\beta a}{\beta!}\eta^\beta \hat{b}_1(\eta)d\eta = \sum_{|\beta|\le N}\frac{(2\pi i)^{-|\beta|}}{\beta!}D_\xi^\beta a D_x^\beta b_1\ .$$

Remainder estimate I

- For $|\alpha| = N+1$,

$$\begin{aligned} &\| \int_0^1 \alpha_{t\eta}(D^\alpha_\beta a)(1-t)^N dt \langle\xi\rangle^{-m-1} \| \\ \le &\int_0^1 \| (D^\alpha_\beta a)\langle\xi\rangle^{-m+1} \| \| \langle \xi + t\eta\rangle^{m+1}\langle\xi\rangle^{-m-1} \| \, dt \\ \le &\int_0^1 (1-t)^N (t|\eta|)^{|m+1|} dt \le A_{N,m}|\eta|^{|m+1|} \end{aligned}$$

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\le &\int_0^1 \| (D^\alpha_\beta a)\langle \xi\rangle^{-m+1} \| \| \langle \xi + t\eta\rangle^{m+1}\langle \xi\rangle^{-m-1} \| \, dt \\
\le &\int_0^1 (1-t)^N (t|\eta|)^{|m+1|} dt \le A_{N,m}|\eta|^{|m+1|}
\end{aligned}$$

- $\| \langle \xi\rangle^{m+1}\hat{b}_1(\eta)\langle \xi\rangle^{-n-m-1} \| \lesssim (1+|\eta|^{-n})$ for any n because $b_1(y)$ is compactly supported with values in S^n. Thus,

$$\begin{aligned}
&\| \int \Big(\int_0^1 \alpha_{t\eta}(D^\alpha_\beta a)(\eta)(1-t)^N dt\Big)\eta^\alpha \hat{b}_1(\eta) d\eta \langle \xi\rangle^{-m-n-1} \| \\
\le &A_{m,N} \int \| \langle \xi\rangle^{m+1}\eta^\alpha \hat{b}_1(\eta)\langle \xi\rangle^{-m-n-1} \| \, d\eta < \infty
\end{aligned}$$

Remainder estimate II

- Take $\alpha_y(b) = b_1(y) + b_2(y)$ s.t. b_2 supported away from 0. Denote $\Delta_\eta = \sum_j D_{\eta_j}^2$ and $\Delta_x = \sum_j D_{x_j}^2$.

$$\begin{aligned} &\int \alpha_\eta(a) b_2(y) e^{2\pi i \eta \cdot y} d\eta dy \\ =& \int \alpha_\eta((\Delta_\xi)^{m_1} a) |y|^{-2m_1} b_2(y) e^{2\pi i \eta \cdot y} d\eta dy \\ =& \int \alpha_\eta((\Delta_\xi)^{m_1} a) \Delta_y^{-2m_2}(|y|^{-2m_1} b_2(y)) |\eta|^{-2m_2} e^{2\pi i \eta \cdot y} d\eta dy \end{aligned}$$

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$$\begin{aligned}
&\int \alpha_\eta(a) b_2(y) e^{2\pi i\eta\cdot y} d\eta dy \\
&= \int \alpha_\eta((\Delta_\xi)^{m_1} a)|y|^{-2m_1} b_2(y) e^{2\pi i\eta\cdot y} d\eta dy \\
&= \int \alpha_\eta((\Delta_\xi)^{m_1} a)\Delta_y^{-2m_2}(|y|^{-2m_1} b_2(y))|\eta|^{-2m_2} e^{2\pi i\eta\cdot y} d\eta dy
\end{aligned}$$

- Note that

$$\begin{aligned}
&\| \alpha_\eta((\Delta_\xi)^{m_1} a)\langle\xi\rangle^{-m+2m_1} \| \lesssim |\eta|^{|-m+2m_1|} \,, \\
&\| \langle\xi\rangle^{m-2m_1} \Delta_y^{-2m_2}(|y|^{-2m_1} b_2(y))\langle\xi\rangle^{-m-2m_1-n} \| \lesssim B_{m,m_1,n}|y|^{-2m_1}
\end{aligned}$$

- Choose m_1, m_2 large enough the integral $\cdot\langle\xi\rangle^{-m-n+N+1}$ converges absolutely.

Remainder estimate II

- Take $\alpha_y(b) = b_1(y) + b_2(y)$ s.t. b_2 supported away from 0. Denote $\Delta_\eta = \sum_j D^2_{\eta_j}$ and $\Delta_x = \sum_j D^2_{x_j}$.

$$\begin{aligned}&\int \alpha_\eta(a) b_2(y) e^{2\pi i \eta\cdot y} d\eta dy \\ =&\int \alpha_\eta((\Delta_\xi)^{m_1} a)|y|^{-2m_1} b_2(y) e^{2\pi i \eta\cdot y} d\eta dy \\ =&\int \alpha_\eta((\Delta_\xi)^{m_1} a)\Delta_y^{-2m_2}(|y|^{-2m_1} b_2(y))|\eta|^{-2m_2} e^{2\pi i \eta\cdot y} d\eta dy\end{aligned}$$

- Note that

$$\begin{aligned}&\| \alpha_\eta((\Delta_\xi)^{m_1} a)\langle\xi\rangle^{-m+2m_1} \| \lesssim |\eta|^{|-m+2m_1|} , \\ &\| \langle\xi\rangle^{m-2m_1} \Delta_y^{-2m_2}(|y|^{-2m_1} b_2(y))\langle\xi\rangle^{-m-2m_1-n} \| \lesssim B_{m,m_1,n}|y|^{-2m_1}\end{aligned}$$

- Choose m_1, m_2 large enough the integral $\cdot\langle\xi\rangle^{-m-n+N+1}$ converges absolutely.
- the estimates are uniform in ϵ. Take $b_\epsilon(y) \to \alpha_y(b)$

Summary and further directions

- quantum Euclidean space with non-commuting $[\xi_j, \xi_k] = -i\theta_{jk}$.
- the definition for noncommutative symbols
- composition identity and L_2-boundedness of 0-order operators.

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- quantum Euclidean space with non-commuting $[\xi_j, \xi_k] = -i\theta_{jk}$.
- the definition for noncommutative symbols
- composition identity and L_2-boundedness of 0-order operators.
- generalizations to Widom's calculus and Getzler's super-symbol calculus.
- applications in calculation of Connes-Moscovici's local index formula. (Sukochev+co in locally compact case)

Thanks for listening!