Interpolation of noncommutative symmetric martingale spaces

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Introduction

The following relationship between Hardy space of noncommutative martingales and its conditioned version holds.

$$H_p^c(\mathcal{M}) = h_p^c(\mathcal{M}) + h_p^d(\mathcal{M}) \tag{0.1}$$

for all $1 \leq p \leq 2$, and

$$H_p^c(\mathcal{M}) = h_p^c(\mathcal{M}) \cap h_p^d(\mathcal{M}) \tag{0.2}$$

for all $2 \leq p < \infty$ (see [1,2])

1. M. Junge and T. Mei, Noncommutative Riesz transforms–a probabilistic approach, *Amer. J. Math.* **132** (2010), 611-680.

2. M. Perrin, A noncommutative Davis' decomposition for martingales, *J. Lond. Math. Soc.* **80** (2009), 627-648.

Introduction

We will present some extensions of (0.1) and (0.2) to the symmetric space case. We prove the following result: Let E be a symmetric Banach spaces on $[0, \infty)$.

1) If E is separable and $1 < p_E \leq q_E < 2$, then

$$H_E^c(\mathcal{M}) = h_E^c(\mathcal{M}) + h_E^d(\mathcal{M}).$$

2) If $2 < p_E \leq q_E < \infty$, then

$$H_E^c(\mathcal{M}) = h_E^c(\mathcal{M}) \cap h_E^d(\mathcal{M}).$$

Introduction

Musat[3] studied the noncommutative BMO and its interpolation properties. She proved noncommutative analogues of the classical interpolation results between BMO and L_p spaces (respectively, Hardy spaces). In [4], the authors considered the interpolation of the conditioned Hardy spaces h_p and presented an extension of Musat's results to the conditioned case.

3. M. Musat, Interpolation between noncommutative BMO and noncommutative L^p-spaces, *J. Funct. Anal.*, **202**(2003), 195-225.

4. T. N. Bekjan, Z. Chen, M. Perrin, Z. Yin, Atomic decomposition and interpolation for Hardy spaces of noncommutative martingales, *J. Funct. Anal.*, **258** (2010), 2483-2505.

Introduction

We will present some extensions of interpolation results in [3,4] to the symmetric space case. We prove the following result: Let $E_1,\ E_2$ be symmetric Banach spaces on $[0,\infty)$.
1) Suppose $E_1,\ E_2$ satisfy the Fatou property, $1 < p_{E_1} \le q_{E_1} < 2$, $1 < p_{E_2} \le q_{E_2} < 2$ and either E_1 or E_2 has order continuous norm. If $0 < \theta < 1$ and $E = (E_1, E_2)_\theta$, then

$$h_E(\mathcal{M}) = (h_{E_0}(\mathcal{M}), h_{E_1}(\mathcal{M}))_\theta$$

holds with equivalent norms.
2) Suppose $E_1,\ E_2$ are fully symmetric Banach spaces, $2 < p_{E_1} \le q_{E_1} < \infty$ and $2 < p_{E_2} \le q_{E_2} < \infty$. If $0 < \theta < 1$ and $E = (E_1, E_2)_\theta$, then

$$h_E(\mathcal{M}) = (h_{E_0}(\mathcal{M}), h_{E_1}(\mathcal{M}))_\theta$$

holds with equivalent norms.

symmetric spaces

Let (Ω, Σ, m) be a σ-measure space and $L(\Omega)$ be the linear space of all measurable, a.e. finite functions on Ω. Define $L_0(\Omega)$ as the subspace of $L(\Omega)$ which consists of all functions x such that $m(\{\omega \in (0,\infty) : |x(\omega)| > s\})$ is finite for some s. Let $x \in L_0(\Omega)$. Recall that the decreasing rearrangement function of x is defined by

$$\mu_t(x) = inf\{s > 0 : m(\{\omega \in \Omega : |x(\omega)| > s\}) \leq t\},\ t > 0.$$

For $x, y \in L_0(\Omega)$ we say x is *majorized* by y, and write $x \preccurlyeq y$, if

$$\int_0^t \mu_s(x)ds \leq \int_0^t \mu_s(y)ds, \quad \text{for all} \quad t > 0.$$

symmetric spaces

Recall the following terminology. A quasi Banach lattice E of measurable functions on $[0,\infty)$ is called a symmetric quasi Banach space on $[0,\infty)$ if E satisfying the following properties: if $f \in E$, $g \in L_0([0,\infty))$ and $\mu(g) \le \mu(f)$ implies that $g \in E$ and $\|g\|_E \le \|f\|_E$.

Let E be a symmetric Banach space on $[0,\infty)$. If for every net $(x_i)_{i\in I}$ in E satisfying $0 \le x_i \uparrow$ and $\sup_{i\in I}\|x_i\|_E < \infty$ the supremum $x = \sup_{i\in I} x_i$ exists in E and $\|x_i\|_E \uparrow \|x\|_E$, we say E has the Fatou property.

symmetric spaces

A symmetric Banach space E on $[0,\infty)$ is called fully symmetric if, in addition, for $x \in L_0([0,\infty))$ and $y \in E$ with $x \preceq y$ it follows that $x \in E$ and $\|x\|_E \leq \|y\|_E$.
The K$\ddot{o}$the dual of a symmetric Banach space E on $[0,\infty)$ is the symmetric Banach space $E^\times$ defined by

$$E^\times = \left\{ x \in L_0([0,\infty)) : \ sup\{ \int_0^\infty |x(t)y(t)|dt \ : \|x\|_E \leq 1\} < \infty \right\};$$

$$\|y\|_{E^\times} = sup\{ \int_0^\infty |x(t)y(t)|dt \ : \|x\|_E \leq 1\}, \quad y \in E^\times.$$

symmetric spaces

A symmetric Banach space E on $[0, \infty)$ is separable if and only if $E = E^{\times}$ isometrically. Moreover, a symmetric Banach space which is separable or has the Fatou property is automatically fully symmetric.

Boyd index

For any $0 < a < \infty$, let the dilation operator D_a on $L_0([0,\infty))$ defined by

$$(D_a f)(s) = f(as)\chi_I(as) \quad (s \in [0,\infty)).$$

If E is a symmetric Banach space on I, then D_a is a bounded linear operator. Define the lower Boyd index p_E of E by

$$p_E = \sup\{p > 0 : \exists c > 0 \, \forall 0 < a \leq 1 \, \|D_a f\|_E \leq c a^{-\frac{1}{p}} \|f\|_E\}$$

and the upper Boyd index q_E of E by

$$q_E = \inf\{q > 0 : \exists c > 0 \, \forall a \geq 1 \, \|D_a f\|_E \leq c a^{-\frac{1}{q}} \|f\|_E\}.$$

It is clear from the definitions that

$$1 \leq p_E \leq q_E \leq \infty.$$

Boyd index

If E is a symmetric Banach space on $[0, \infty)$, then

$$\frac{1}{p_E} + \frac{1}{q_{E^\times}} = 1, \quad \frac{1}{p_{E^\times}} + \frac{1}{q_E} = 1. \tag{1.1}$$

For more details on symmetric Banach space we refer to [5,6].

5. S.G. Krein, J.I. Petunin, E.M. Semenov, Interpolation of linear operators, *Translations of Mathematical Monographs*, vol.54, AMS, 1982.

6. J. Lindentrauss, L. Tzafriri, *Classical Banach space*II, Springer-Verlag, Berlin, 1979.

noncommutative symmetric spaces

Throughout this talk, we denote by $\mathcal{M}$ a semi-finite von Neumann algebra on the Hilbert space $\mathcal{H}$ with a faithful normal semi-finite trace τ. We denote by $L_0(\mathcal{M})$ the linear space of all τ-measurable operators. One can show that $L_0(\mathcal{M})$ is a $*$-algebra with sum and product being the respective closure of the algebraic sum and product.

Let $x \in L_0(\mathcal{M})$, and let $e_s^\perp(|x|) = 1_{(s,\infty)}(x)$ is the spectral projection of $|x|$ corresponding to the interval (s, ∞). Define

$$\lambda_s(x) = \tau\big(e_s^\perp(|x|)\big), \quad s > 0$$

and

$$\mu_t(x) = \inf\{s > 0 \ : \ \lambda_s(x) \leq t\}, \quad t > 0.$$

The function $s \mapsto \lambda_s(x)$ is called the distribution function of x and the $\mu_t(x)$ the generalized singular numbers (decreasing rearrangement) of x. For more details on generalized singular value function of measurable operators we refer to [7].

7. T. Fack and H. Kosaki, Generalized s-numbers of τ-measurable operators, *Pac. J. Math.* **123** (1986), 269-300.

noncommutative symmetric spaces

Let E be a symmetric Banach space on $[0, \infty)$. We define

$$L_E(\mathcal{M}) = \{x \in L_0(\mathcal{M}) : \ \mu_{\cdot}(x) \in E\};$$

$$\|x\|_{L_E(\mathcal{M})} = \|\mu_{\cdot}(x)\|_E, \qquad x \in L_E(\mathcal{M}).$$

Then $(L_E(\mathcal{M}), \|.\|_{L_E(\mathcal{M})})$ is a Banach space (cf. [8,9]).

8. F. Sukochev, Completeness of quasi-normed symmetric operator spaces, *Indag. Math.* (N.S.) **25**(2) (2014), 376-388.

9. Q. Xu, Analytic functions with values in lattices and symmetric spaces of measurable operators, *Math. Proc. Camb. Phil. Soc.* **109** (1991),541-563.

In what follows, unless otherwise specified, we always denote by E a symmetric Banach space on $[0,\infty)$.Let $a=(a_n)_{n\geq 0}$ be a finite sequence in $L_E(\mathcal{M})$, define

$$\|a\|_{L_E(\mathcal{M},\ell_c^2)} = \|(\sum_{n\geq 0}|a_n|^2)^{1/2}\|_{L_E(\mathcal{M})},$$

$$\|a\|_{L_E(\mathcal{M},\ell_r^2)} = \|(\sum_{n\geq 0}|a_n^*|^2)^{1/2}\|_{L_E(\mathcal{M})}.$$

This gives two noms on the family of all finite sequences in $L_E(\mathcal{M})$. To see this, denoting by $\mathcal{B}(\ell_2)$ the algebra of all bounded operators on ℓ_2 with its usual trace tr, let us consider the von Neumann algebra tensor product $\mathcal{M}\otimes\mathcal{B}(\ell_2)$ with the product trace $\tau\otimes tr$. $\tau\otimes tr$ is a semi-finite normal faithful trace, the associated noncommutative L_E space is denoted by $L_E(\mathcal{M}\otimes\mathcal{B}(\ell_2))$.

Now, any finite sequence $a = (a_n)_{n\geq 1}$ in $L_E(\mathcal{M})$ can be regarded as an element in $L_E(\mathcal{M} \otimes \mathcal{B}(\ell_2))$ via the following map

$$a \longmapsto T(a) = \begin{pmatrix} a_1 & 0 & \dots \\ a_2 & 0 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

that is, the matrix of $T(a)$ has all vanishing entries except those in the first column which are the a_n's. Such a matrix is called a column matrix, and the closure in $L_E(\mathcal{M} \otimes \mathcal{B}(\ell_2))$ of all column matrices is called the column subspace of $L_E(\mathcal{M} \otimes \mathcal{B}(\ell_2))$. Then

$$\|a\|_{L_E(\mathcal{M},\ell_c^2)} = \||T(a)|\|_{L_E(\mathcal{M}\otimes\mathcal{B}(\ell_2))} = \|T(a)\|_{L_E(\mathcal{M}\otimes\mathcal{B}(\ell_2))}.$$

Therefore, $\|.\|_{L_E(\mathcal{M},\ell_c^2)}$ defines a norm on the family of all finite sequences of $L_E(\mathcal{M})$. The corresponding completion is a Banach space, denoted by $L_E(\mathcal{M}, \ell_c^2)$.

Similarly, we may show that $\|.\|_{L_E(\mathcal{M},\ell_r^2)}$ is a norm on the family of all finite sequence in $L_E(\mathcal{M})$. As above, it defines a Banach space $L_E(\mathcal{M},\ell_r^2)$, which now is isometric to the row subspace of $L_E(\mathcal{M}\otimes\mathcal{B}(\ell_2))$ consisting of matrices whose nonzero entries lie only in the first row.

We also need $L_E^d(\mathcal{M})$, the space of all sequences $a=(a_n)_{n\geq 1}$ in $L_E(\mathcal{M})$ such that

$$\|a\|_{L_E^d(\mathcal{M})}=\|diag(a_n)\|_{L_E(\mathcal{M}\otimes\mathcal{B}(\ell_2))}<\infty,$$

where $diag(a_n)$ is the matrix with the a_n on the diagonal and zeroes elsewhere.

Let us now recall the general setup for noncommutative martingales. In the sequel, we always denote by $(\mathcal{M}_n)_{n\geq 1}$ an increasing sequence of von Neumann subalgebras of $\mathcal{M}$ whose union $\cup_{n\geq 1}\mathcal{M}_n$ generates $\mathcal{M}$ (in the w^*-topology). $(\mathcal{M}_n)_{n\geq 1}$ is called a filtration of $\mathcal{M}$. For $n \geq 1$, we assume that there exists a trace preserving conditional expectation $\mathcal{E}_n$ from $\mathcal{M}$ onto $\mathcal{M}_n$. The restriction of τ to $\mathcal{M}_n$ is still denoted by τ. It is well-known that $\mathcal{E}_n$ extends to a contractive projection from $L_p(\mathcal{M}, \tau)$ onto $L_p(\mathcal{M}_n, \tau_n)$ for all $1 \leq p \leq \infty$. More generally, if E is a symmetric Banach function space on $[0, \infty)$ then $\mathcal{E}_n$ is a contraction from $L_E(\mathcal{M}, \tau)$ onto $L_E(\mathcal{M}_n, \tau)$.

noncommutative martingale

A noncommutative L_E-martingale with respect to $(\mathcal{M}_n)_{n\geq 1}$ is a sequence $x = (x_n)_{n\geq 1}$ such that $x_n \in L_E(\mathcal{M}_n)$ and

$$\mathcal{E}_n(x_{n+1}) = x_n$$

for any $n \geq 1$. Set $\|x\|_{L_E(\mathcal{M})} = \sup_{n\geq 1} \|x_n\|_{L_E(\mathcal{M})}$. If $\|x\|_{L_E(\mathcal{M})} < \infty$, then x is said to be a bounded L_E-martingale. Let x be a noncommutative martingale. The martingale difference sequence of x, denoted by $dx = (dx_n)_{n\geq 1}$, is defined as

$$dx_1 = x_1, \quad dx_n = x_n - x_{n-1}, \quad n \geq 2.$$

snoncommutative martingale spaces

For any finite martingale $x = (x_n)_{n\geq 1}$ in $L_E(M)$, we set

$$S^c(x) = \Big(\sum_{k\geq 1} |dx_k|^2\Big)^{\frac{1}{2}} \quad \text{and} \quad S^r(x) = \Big(\sum_{k\geq 1} |dx_k^*|^2\Big)^{\frac{1}{2}},$$

and define

$$\|x\|_{H_E^c(\mathcal{M})} = \|S^c(x)\|_{L_E(\mathcal{M})} = \|dx\|_{L_E(\mathcal{M},\ell_c^2)}$$

$$\big(\text{resp. } \|x\|_{H_E^r(\mathcal{M})} = \|S^r(x)\|_{L_E(\mathcal{M})} = \|dx\|_{L_E(\mathcal{M},\ell_r^2)}\big).$$

Let $H_E^c(\mathcal{M})$ be (resp. $H_E^r(\mathcal{M})$) the corresponding completion. Then $H_E^c(\mathcal{M})$ (resp. $H_E^r(\mathcal{M})$) is Banach space.

noncommutative martingale spaces

We now consider the conditioned versions of square functions and Hardy spaces developed in [10]. Let $a = (a_n)_{n\geq 1}$ be a finite sequence in $\mathcal{M}$. We define (recalling $\mathcal{E}_0 = \mathcal{E}_1$)

$$\|a\|_{L_E^{cond}(\mathcal{M},\ell_c^2)} = \|\Big(\sum_{n\geq 1} \mathcal{E}_{n-1}(|a_n|^2)\Big)^{\frac{1}{2}}\|_{L_E(\mathcal{M})}.$$

It is shown in [12] that $\|\cdot\|_{L_E^{cond}(\mathcal{M},\ell_c^2)}$ is a norm on the vector space of all finite sequences in $\mathcal{M}$. We define $L_E^{cond}(\mathcal{M},\ell_c^2)$ is the corresponding completion. Similarly, we define the conditioned row space $L_E^{cond}(\mathcal{M},\ell_r^2)$. Note that $L_E^{cond}(\mathcal{M},\ell_c^2)$ (resp. $L_E^{cond}(\mathcal{M},\ell_r^2)$) is the conditioned version of $L_E(\mathcal{M},\ell_c^2)$ (resp. $L_E(\mathcal{M},\ell_r^2)$). The space $L_E^{cond}(\mathcal{M},\ell_c^2)$ (resp. $L_E^{cond}(\mathcal{M},\ell_r^2)$) can be viewed as a closed subspace of the column (resp. row) subspace of $L_E(\mathcal{M}\otimes\mathcal{B}(\ell_2(\mathbb{N}^2)))$. We refer to [10,11,12] for more details on this.

10. M. Junge, Q. Xu, Noncommutative Burkholder/Rosenthal inequalities, *Ann. Probab.*, **31** (2003), 948-995.

11. M. Junge, Doob's inequality for noncommutative martingales, *J. Reine Angew. Math.* **549** (2002), 149-190.

12. N. Randrianantoanina and L. Wu, Martingale inequalities in noncommutative symmetric spaces, *J. Funct. Anal.* **269** (2015), 2222-2253.

noncommutative martingale spaces

For a finite noncommutative L_E-martingale $x = (x_n)_{n\geq 1}$ define (with $\mathcal{E}_0 = \mathcal{E}_1$)

$$\|x\|_{h_E^c(\mathcal{M})} = \|dx\|_{L_E^{cond}(\mathcal{M},\ell_c^2)} \quad \text{and} \quad \|x\|_{h_E^r(\mathcal{M})} = \|dx\|_{L_E^{cond}(\mathcal{M},\ell_r^2)}.$$

Let $h_E^c(\mathcal{M})$ and $h_E^r(\mathcal{M})$ be the corresponding completions. Then $h_E^c(\mathcal{M})$ and $h_E^r(\mathcal{M})$ are Banach spaces. We define the column and row conditioned square functions as follows. For any finite martingale $x = (x_n)_{n\geq 1}$ in $L_E(M)$, we set

$$s^c(x) = \Big(\sum_{k\geq 1} \mathcal{E}_{k-1}(|dx_k|^2)\Big)^{\frac{1}{2}} \quad \text{and} \quad s^r(x) = \Big(\sum_{k\geq 1} \mathcal{E}_{k-1}(|dx_k^*|^2)\Big)^{\frac{1}{2}}.$$

Then

$$\|x\|_{h_E^c(\mathcal{M})} = \|s^c(x)\|_{L_E(\mathcal{M})} \quad \text{and} \quad \|x\|_{h_E^r(\mathcal{M})} = \|s^r(x)\|_{L_E(\mathcal{M})}.$$

noncommutative martingale spaces

Let $x = (x_n)_{n\geq 0}$ be a finite L_E-martingale, we set

$$s^d(x) = diag(|dx_n|)$$

We note that

$$\|s^d(x)\|_{L_E(\mathcal{M}\otimes\mathcal{B}(\ell_2))} = \|dx_n\|_{L_E^d(\mathcal{M})}$$

Let $h_E^d(\mathcal{M})$ be the subspace of $L_E^d(\mathcal{M})$ consisting of all martingale difference sequences.

noncommutative martingale spaces

We define the Hardy space of noncommutative martingales and its conditioned version as follows. For $1 \le p_E \le q_E < 2$,

$$H_E(\mathcal{M}) = H_E^c(\mathcal{M}) + H_E^r(\mathcal{M}),$$

equipped with the norm

$$\|x\|_{H_E(\mathcal{M})} = \inf \left\{ \begin{array}{c} \|y\|_{H_E^c(\mathcal{M})} + \|z\|_{H_E^r(\mathcal{M})} : \\ x = y + z,\ y \in H_E^c(\mathcal{M}),\ z \in H_E^r(\mathcal{M})) \end{array} \right\}$$

and

$$h_E(\mathcal{M}) = h_E^c(\mathcal{M}) + h_E^r(\mathcal{M}) + h_E^d(\mathcal{M}),$$

equipped with the norm

$$\|x\|_{h_E(\mathcal{M})} = \inf \left\{ \begin{array}{c} \|y\|_{h_E^c(\mathcal{M})} + \|z\|_{h_E^r(\mathcal{M})} + \|w\|_{h_E^d(\mathcal{M})} : \\ x = y + z + w,\ y \in h_E^c(\mathcal{M}), \\ z \in h_E^r(\mathcal{M})),\ w \in h_E^d(\mathcal{M}) \end{array} \right\}.$$

noncommutative martingale spaces

For $2 \leq p_E \leq q_E < \infty$,

$$H_E(\mathcal{M}) = H_E^c(\mathcal{M}) \cap H_E^r(\mathcal{M}),$$

equipped with the norm

$$\|x\| = \max\{\|x\|_{H_E^c(\mathcal{M})},\ \|x\|_{H_E^r(\mathcal{M})}\}.$$

and

$$h_E(\mathcal{M}) = h_E^c(\mathcal{M}) \cap h_E^r(\mathcal{M}) \cap h_E^d(\mathcal{M}),$$

equipped with the norm

$$\|x\|_{h_E(\mathcal{M})} = \max\{\|x\|_{h_E^c(\mathcal{M})},\ \|x\|_{h_E^r(\mathcal{M})},\ \|x\|_{h_E^d(\mathcal{M})}\}.$$

Using Lemma 6.4 in [10], Theorem 2.3 in [13] and Theorem 3.4 in [14] we obtain the following result.

Proposition

Let E be a fully symmetric Banach space on $[0,\infty)$ with $1 < p_E \leq q_E < \infty$. Then we have

(i) $h_E^c(\mathcal{M})$, $h_E^r(\mathcal{M})$ *are complemented in* $L_E(\mathcal{M} \otimes \mathcal{B}(\ell_2(\mathbb{N}^2)))$.

(ii) $H_E^c(\mathcal{M})$, $H_E^r(\mathcal{M})$ *are complemented in* $L_E(\mathcal{M} \otimes \mathcal{B}(\ell_2))$.

13. G. Pisier, Q. Xu, Non-commutative martingale inequalities, *Comm. Math. Phys.* **189** (1997), 667-698.

14. P. G. Dodds, T. K. Dodds, B. de Pager, Fully symmetric operator spaces, *Inter Equat OperTh.*, **15** (1992), 942-972.

By Theorem 5.6 in [15] and the above proposition, it follows that

Theorem

Let E be a separable symmetric Banach space on $[0,\infty)$ with $1 < p_E \leq q_E < \infty$. Then we have

(i) $(h_E^c(\mathcal{M}))^* = h_{E^\times}^c(\mathcal{M})$ *with equivalent norms.*

(ii) $(H_E^c(\mathcal{M}))^* = H_{E^\times}^c(\mathcal{M})$ *with equivalent norms.*

Similarly, $(h_E^r(\mathcal{M}))^* = h_{E^\times}^r(\mathcal{M})$ *and* $(H_E^r(\mathcal{M}))^* = H_{E^\times}^r(\mathcal{M})$ *with equivalent norms.*

15. P. G. Dodds, T. K. Dodds, B. de Pager, Noncommutative Köthe duality, *Trans. Amer. Math. Soc.*, **339** (1993) 717-750.

noncommutative martingale spaces

Proposition

Let E *be a symmetric Banach space on* $[0,\infty)$ *with* $2 < p_E \leq q_E < \infty$*. Then we have*

(i) $H_E^c(\mathcal{M})) = h_E^c(\mathcal{M}) \cap h_E^d(\mathcal{M})$ *with equivalent norms.*

(ii) $H_E^r(\mathcal{M}) = H_E^r(\mathcal{M}) \cap h_E^d(\mathcal{M})$ *with equivalent norms.*

noncommutative martingale spaces

Proposition

Let E be a symmetric Banach space on $[0,\infty)$. If E is separable or the dual of a separable space and satisfies $1 < p_E \leq q_E < 2$. Then we have

(i) $H^c_E(\mathcal{M})) = h^c_E(\mathcal{M}) + h^d_E(\mathcal{M})$ *with equivalent norms.*

(ii) $H^r_E(\mathcal{M}) = h^r_E(\mathcal{M}) + h^d_E(\mathcal{M})$ *with equivalent norms.*

Corollary

Let E be a separable symmetric Banach space on $[0,\infty)$ with $1 < p_E \leq q_E < 2$. Then we have

$$L_E(\mathcal{M}) = h_E^c(\mathcal{M}) + h_E^d(\mathcal{M}) + h_E^r(\mathcal{M}).$$

noncommutative martingale spaces

Theorem

Let E be a symmetric Banach space on $[0,\infty)$. If E is separable or the dual of a separable space and satisfies $1 < p_E \leq q_E < 2$. Then we have

$$(h_E(\mathcal{M}))^* = h_{E^\times}(\mathcal{M})$$

with equivalent norms.

Interpolation

We recall interpolation of noncommutative symmetric spaces. Let $E_1,\ E_2$ be fully symmetric spaces on $[0,\infty)$ and $0<\theta<1$. If E is complex interpolation of E_1 and E_2, i.e. $E=(E_1,E_2)_\theta$. Then

$$L_E(\mathcal{M}) = (L_{E_1}(\mathcal{M}), L_{E_2}(\mathcal{M}))_\theta. \tag{3.1}$$

Since $\{diag(a_n): \quad (a_n)\subset\mathcal{M}\}$ is von Neumann subalgebra of $\mathcal{M}\otimes\mathcal{B}(\ell_2)$, we have that

$$L_E^d(\mathcal{M})) = (L_{E_1}^d(\mathcal{M})), L_{E_2}^d(\mathcal{M})))_\theta. \tag{3.2}$$

For more details on interpolation of noncommutative symmetric spaces we refer to [14].

Interpolation

Proposition

Let E_j be a fully symmetric Banach spaces on $[0,\infty)$ with $1 < p_{E_j} \leq q_{E_j} < 2$ $(j = 1, 2)$. If $0 < \theta < 1$ and $E = (E_1, E_2)_\theta$, then

(i)

$$h_E^c(\mathcal{M}) = (h_{E_1}^c(\mathcal{M}), h_{E_2}^c(\mathcal{M}))_\theta,$$

$$h_E^r(\mathcal{M}) = (h_{E_1}^r(\mathcal{M}), h_{E_2}^r(\mathcal{M}))_\theta.$$

(ii)

$$H_E^c(\mathcal{M}) = (H_{E_1}^c(\mathcal{M}), H_{E_2}^c(\mathcal{M}))_\theta,$$

$$H_E^r(\mathcal{M}) = (H_{E_1}^r(\mathcal{M}), H_{E_2}^r(\mathcal{M}))_\theta.$$

Interpolation

Lemma

Let E_j be a symmetric Banach space on $[0,\infty)$ with $1 < p_{E_j} \leq q_{E_j} < \infty$ $(j = 1, 2)$ and $0 < \theta < 1$. Suppose that $E = (E_1, E_2)_\theta$, then the following holds with equivalent norms

$$(h^d_{E_1}(\mathcal{M}), h^d_{E_2}(\mathcal{M}))_\theta = h^d_E(\mathcal{M}).$$

Interpolation

Theorem

Let E_j be a symmetric Banach space on $[0, \infty)$ satisfying the Fatou property, and let $1 < p_{E_j} \leq q_{E_j} < 2$ $(j = 1, 2)$. Suppose that either E_1 or E_2 has order continuous norm. If $0 < \theta < 1$ and $E = (E_1, E_2)_\theta$, then

(i) $h_E(\mathcal{M}) = (h_{E_1}(\mathcal{M}), h_{E_2}(\mathcal{M}))_\theta$ *holds with equivalent norms.*

(ii) $H_E(\mathcal{M}) = (H_{E_1}(\mathcal{M}), H_{E_2}(\mathcal{M}))_\theta$ *holds with equivalent norms.*

Interpolation

Theorem

Let E_j *be a fully symmetric Banach space on* $[0,\infty)$ *with* $2 < p_{E_j} \leq q_{E_j} < \infty$ $(j = 1, 2)$. *If* $0 < \theta < 1$ *and* $E = (E_1, E_2)_\theta$, *then*

(i) $h_E(\mathcal{M}) = (h_{E_1}(\mathcal{M}), h_{E_2}(\mathcal{M}))_\theta$ *holds with equivalent norms.*

(ii) $H_E(\mathcal{M}) = (H_{E_1}(\mathcal{M}), H_{E_2}(\mathcal{M}))_\theta$ *holds with equivalent norms.*

main references

- T.N. Bekjan, *Interpolation of noncommutative symmetric martingale spaces* J. Operator Theory **7**7 (2017), 245–259.
- T. N. Bekjan, Duality for symmetric Hardy spaces of noncommutative martingales, arXiv:1510.03967.

Thanks for your attention!