Interpolation of noncommutative symmetric martingale spaces

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The following relationship between Hardy space of noncommutative martingales and its conditioned version holds.

\[ H^c_p(\mathcal{M}) = h^c_p(\mathcal{M}) + h^d_p(\mathcal{M}) \]  

(0.1)

for all \( 1 \leq p \leq 2 \), and

\[ H^c_p(\mathcal{M}) = h^c_p(\mathcal{M}) \cap h^d_p(\mathcal{M}) \]  

(0.2)

for all \( 2 \leq p < \infty \) (see [1,2])


We will present some extensions of (0.1) and (0.2) to the symmetric space case. We prove the following result: Let $E$ be a symmetric Banach spaces on $[0, \infty)$.

1) If $E$ is separable and $1 < p_E \leq q_E < 2$, then

$$H^c_E(\mathcal{M}) = h^c_E(\mathcal{M}) + h^d_E(\mathcal{M}).$$

2) If $2 < p_E \leq q_E < \infty$, then

$$H^c_E(\mathcal{M}) = h^c_E(\mathcal{M}) \cap h^d_E(\mathcal{M}).$$
Musat[3] studied the noncommutative $BMO$ and its interpolation properties. She proved noncommutative analogues of the classical interpolation results between $BMO$ and $L_p$ spaces (respectively, Hardy spaces). In [4], the authors considered the interpolation of the conditioned Hardy spaces $h_p$ and presented an extension of Musat’s results to the conditioned case.


We will present some extensions of interpolation results in [3,4] to the symmetric space case. We prove the following result: Let $E_1$, $E_2$ be symmetric Banach spaces on $[0, \infty)$. 

1) Suppose $E_1$, $E_2$ satisfy the Fatou property, $1 < p_{E_1} \leq q_{E_1} < 2$, $1 < p_{E_2} \leq q_{E_2} < 2$ and either $E_1$ or $E_2$ has order continuous norm. If $0 < \theta < 1$ and $E = (E_1, E_2)_\theta$, then 

$$h_E(\mathcal{M}) = (h_{E_0}(\mathcal{M}), h_{E_1}(\mathcal{M}))_\theta$$

holds with equivalent norms.

2) Suppose $E_1$, $E_2$ are fully symmetric Banach spaces, $2 < p_{E_1} \leq q_{E_1} < \infty$ and $2 < p_{E_2} \leq q_{E_2} < \infty$. If $0 < \theta < 1$ and $E = (E_1, E_2)_\theta$, then 

$$h_E(\mathcal{M}) = (h_{E_0}(\mathcal{M}), h_{E_1}(\mathcal{M}))_\theta$$

holds with equivalent norms.
Let \((\Omega, \Sigma, m)\) be a \(\sigma\)-measure space and \(L(\Omega)\) be the linear space of all measurable, a.e. finite functions on \(\Omega\). Define \(L_0(\Omega)\) as the subspace of \(L(\Omega)\) which consists of all functions \(x\) such that \(m(\{\omega \in (0, \infty) : |x(\omega)| > s\})\) is finite for some \(s\). Let \(x \in L_0(\Omega)\). Recall that the decreasing rearrangement function of \(x\) is defined by

\[
\mu_t(x) = \inf \{s > 0 : m(\{\omega \in \Omega : |x(\omega)| > s\}) \leq t\}, \; t > 0.
\]

For \(x, y \in L_0(\Omega)\) we say \(x\) is majorized by \(y\), and write \(x \preceq y\), if

\[
\int_0^t \mu_s(x) ds \leq \int_0^t \mu_s(y) ds, \quad \text{for all} \quad t > 0.
\]
Recall the following terminology. A quasi Banach lattice $E$ of measurable functions on $[0, \infty)$ is called a symmetric quasi Banach space on $[0, \infty)$ if $E$ satisfying the following properties: if $f \in E$, $g \in L_0([0, \infty))$ and $\mu(g) \leq \mu(f)$ implies that $g \in E$ and $\|g\|_E \leq \|f\|_E$.

Let $E$ be a symmetric Banach space on $[0, \infty)$. If for every net $(x_i)_{i \in I}$ in $E$ satisfying $0 \leq x_i \uparrow$ and $\sup_{i \in I} \|x_i\|_E < \infty$ the supremum $x = \sup_{i \in I} x_i$ exists in $E$ and $\|x_i\|_E \uparrow \|x\|_E$, we say $E$ has the Fatou property.
A symmetric Banach space $E$ on $[0, \infty)$ is called fully symmetric if, in addition, for $x \in L_0([0, \infty))$ and $y \in E$ with $x \preceq y$ it follows that $x \in E$ and $\|x\|_E \leq \|y\|_E$.

The Köthe dual of a symmetric Banach space $E$ on $[0, \infty)$ is the symmetric Banach space $E^\times$ defined by

$$E^\times = \left\{ x \in L_0([0, \infty)) : \sup \{ \int_0^\infty |x(t)y(t)| \, dt : \|x\|_E \leq 1 \} < \infty \right\};$$

$$\|y\|_{E^\times} = \sup \{ \int_0^\infty |x(t)y(t)| \, dt : \|x\|_E \leq 1 \}, \quad y \in E^\times.$$
A symmetric Banach space $E$ on $[0, \infty)$ is separable if and only if $E = E^\times$ isometrically. Moreover, a symmetric Banach space which is separable or has the Fatou property is automatically fully symmetric.
For any $0 < a < \infty$, let the dilation operator $D_a$ on $L_0([0, \infty))$ defined by

$$(D_a f)(s) = f(as) \chi_I(as) \quad (s \in [0, \infty)).$$

If $E$ is a symmetric Banach space on $I$, then $D_a$ is a bounded linear operator. Define the lower Boyd index $p_E$ of $E$ by

$$p_E = \sup\{p > 0 : \exists c > 0 \forall 0 < a \leq 1 \|D_a f\|_E \leq ca^{-\frac{1}{p}}\|f\|_E\}$$

and the upper Boyd index $q_E$ of $E$ by

$$q_E = \inf\{q > 0 : \exists c > 0 \forall a \geq 1 \|D_a f\|_E \leq ca^{-\frac{1}{q}}\|f\|_E\}.$$  

It is clear from the definitions that

$$1 \leq p_E \leq q_E \leq \infty.$$
If $E$ is a symmetric Banach space on $[0, \infty)$, then

\[
\frac{1}{p_E} + \frac{1}{q_E^\times} = 1, \quad \frac{1}{p_E^\times} + \frac{1}{q_E} = 1. \quad (1.1)
\]

For more details on symmetric Banach space we refer to [5,6].


Throughout this talk, we denote by $\mathcal{M}$ a semi-finite von Neumann algebra on the Hilbert space $\mathcal{H}$ with a faithful normal semi-finite trace $\tau$. We denote by $L_0(\mathcal{M})$ the linear space of all $\tau$-measurable operators. One can show that $L_0(\mathcal{M})$ is a $*$-algebra with sum and product being the respective closure of the algebraic sum and product.
Let $x \in L_0(\mathcal{M})$, and let $e_s^\perp(|x|) = 1_{(s,\infty)}(x)$ is the spectral projection of $|x|$ corresponding to the interval $(s, \infty)$. Define

$$\lambda_s(x) = \tau(e_s^\perp(|x|)), \quad s > 0$$

and

$$\mu_t(x) = \inf\{s > 0 : \lambda_s(x) \leq t\}, \quad t > 0.$$ 

The function $s \mapsto \lambda_s(x)$ is called the distribution function of $x$ and the $\mu_t(x)$ the generalized singular numbers (decreasing rearrangement) of $x$. For more details on generalized singular value function of measurable operators we refer to [7].

Let $E$ be a symmetric Banach space on $[0, \infty)$. We define

$$L_E(\mathcal{M}) = \{x \in L_0(\mathcal{M}) : \mu.(x) \in E\};$$

$$\|x\|_{L_E(\mathcal{M})} = \|\mu.(x)\|_E, \quad x \in L_E(\mathcal{M}).$$

Then $(L_E(\mathcal{M}), \|\cdot\|_{L_E(\mathcal{M})})$ is a Banach space (cf. [8,9]).


In what follows, unless otherwise specified, we always denote by $E$ a symmetric Banach space on $[0, \infty)$. Let $a = (a_n)_{n \geq 0}$ be a finite sequence in $L_E(M)$, define

$$\|a\|_{L_E(M, \ell^2_c)} = \left\| \left( \sum_{n \geq 0} |a_n|^2 \right)^{1/2} \right\|_{L_E(M)},$$

$$\|a\|_{L_E(M, \ell^2_r)} = \left\| \left( \sum_{n \geq 0} |a_n^*|^2 \right)^{1/2} \right\|_{L_E(M)}.$$

This gives two norms on the family of all finite sequences in $L_E(M)$. To see this, denoting by $\mathcal{B}(\ell_2)$ the algebra of all bounded operators on $\ell_2$ with its usual trace $tr$, let us consider the von Neumann algebra tensor product $M \otimes \mathcal{B}(\ell_2)$ with the product trace $\tau \otimes tr$. $\tau \otimes tr$ is a semi-finite normal faithful trace, the associated noncommutative $L_E$ space is denoted by $L_E(M \otimes \mathcal{B}(\ell_2))$. 
Now, any finite sequence \( a = (a_n)_{n \geq 1} \) in \( L_E(\mathcal{M}) \) can be regarded as an element in \( L_E(\mathcal{M} \otimes \mathcal{B}(\ell_2)) \) via the following map

\[
a \mapsto T(a) = \begin{pmatrix} a_1 & 0 & \cdots \\ a_2 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix},
\]

that is, the matrix of \( T(a) \) has all vanishing entries except those in the first column which are the \( a_n \)'s. Such a matrix is called a column matrix, and the closure in \( L_E(\mathcal{M} \otimes \mathcal{B}(\ell_2)) \) of all column matrices is called the column subspace of \( L_E(\mathcal{M} \otimes \mathcal{B}(\ell_2)) \). Then

\[
\|a\|_{L_E(\mathcal{M},\ell_2^c)} = \|T(a)\|_{L_E(\mathcal{M} \otimes \mathcal{B}(\ell_2))} = \|T(a)\|_{L_E(\mathcal{M} \otimes \mathcal{B}(\ell_2))}.
\]

Therefore, \( \|\cdot\|_{L_E(\mathcal{M},\ell_2^c)} \) defines a norm on the family of all finite sequences of \( L_E(\mathcal{M}) \). The corresponding completion is a Banach space, denoted by \( L_E(\mathcal{M}, \ell_2^c) \).
Similarly, we may show that $\| \cdot \|_{L_E(M, \ell^2_\tau)}$ is a norm on the family of all finite sequence in $L_E(M)$. As above, it defines a Banach space $L_E(M, \ell^2_\tau)$, which now is isometric to the row subspace of $L_E(M \otimes \mathcal{B}(\ell_2))$ consisting of matrices whose nonzero entries lie only in the first row.

We also need $L^d_E(M)$, the space of all sequences $a = (a_n)_{n \geq 1}$ in $L_E(M)$ such that

$$\|a\|_{L_E(M)} = \| \text{diag}(a_n) \|_{L_E(M \otimes \mathcal{B}(\ell_2))} < \infty,$$

where $\text{diag}(a_n)$ is the matrix with the $a_n$ on the diagonal and zeroes elsewhere.
Let us now recall the general setup for noncommutative martingales. In the sequel, we always denote by \((M_n)_{n \geq 1}\) an increasing sequence of von Neumann subalgebras of \(M\) whose union \(\bigcup_{n \geq 1} M_n\) generates \(M\) (in the \(w^*\)-topology). \((M_n)_{n \geq 1}\) is called a filtration of \(M\). For \(n \geq 1\), we assume that there exists a trace preserving conditional expectation \(E_n\) from \(M\) onto \(M_n\). The restriction of \(\tau\) to \(M_n\) is still denoted by \(\tau\). It is well-known that \(E_n\) extends to a contractive projection from \(L_p(M, \tau)\) onto \(L_p(M_n, \tau_n)\) for all \(1 \leq p \leq \infty\). More generally, if \(E\) is a symmetric Banach function space on \([0, \infty)\) then \(E_n\) is a contraction from \(L_E(M, \tau)\) onto \(L_E(M_n, \tau)\).
A noncommutative $L_E$-martingale with respect to $(\mathcal{M}_n)_{n \geq 1}$ is a sequence $x = (x_n)_{n \geq 1}$ such that $x_n \in L_E(\mathcal{M}_n)$ and

$$\mathcal{E}_n(x_{n+1}) = x_n$$

for any $n \geq 1$. Set $\|x\|_{L_E(\mathcal{M})} = \sup_{n \geq 1} \|x_n\|_{L_E(\mathcal{M})}$. If $\|x\|_{L_E(\mathcal{M})} < \infty$, then $x$ is said to be a bounded $L_E$-martingale.

Let $x$ be a noncommutative martingale. The martingale difference sequence of $x$, denoted by $dx = (dx_n)_{n \geq 1}$, is defined as

$$dx_1 = x_1, \quad dx_n = x_n - x_{n-1}, \quad n \geq 2.$$
For any finite martingale $x = (x_n)_{n \geq 1}$ in $L_E(M)$, we set

$$S^c(x) = \left( \sum_{k \geq 1} |dx_k|^2 \right)^{\frac{1}{2}}$$

and

$$S^r(x) = \left( \sum_{k \geq 1} |dx^*_k|^2 \right)^{\frac{1}{2}},$$

and define

$$\|x\|_{H^c_E(M)} = \|S^c(x)\|_{L_E(M)} = \|dx\|_{L_E(M, \ell^2_c)}$$

and

$$\text{(resp. } \|x\|_{H^r_E(M)} = \|S^r(x)\|_{L_E(M)} = \|dx\|_{L_E(M, \ell^2_r)}).$$

Let $H^c_E(M)$ be (resp. $H^r_E(M)$) the corresponding completion. Then $H^c_E(M)$ (resp. $H^r_E(M)$) is Banach space.
We now consider the conditioned versions of square functions and Hardy spaces developed in [10]. Let \( a = (a_n)_{n \geq 1} \) be a finite sequence in \( \mathcal{M} \). We define (recalling \( \mathcal{E}_0 = \mathcal{E}_1 \))

\[
\| a \|_{L^{\text{cond}}_E(\mathcal{M}, \ell^2_\ell)} = \| \left( \sum_{n \geq 1} \mathcal{E}_{n-1} (|a_n|^2) \right)^{1/2} \|_{L_E(\mathcal{M})}.
\]

It is shown in [12] that \( \| \cdot \|_{L^{\text{cond}}_E(\mathcal{M}, \ell^2_\ell)} \) is a norm on the vector space of all finite sequences in \( \mathcal{M} \). We define \( L^{\text{cond}}_E(\mathcal{M}, \ell^2_\ell) \) is the corresponding completion. Similarly, we define the conditioned row space \( L^{\text{cond}}_E(\mathcal{M}, \ell^2_r) \). Note that \( L^{\text{cond}}_E(\mathcal{M}, \ell^2_\ell) \) (resp. \( L^{\text{cond}}_E(\mathcal{M}, \ell^2_r) \)) is the conditioned version of \( L_E(\mathcal{M}, \ell^2_\ell) \) (resp. \( L_E(\mathcal{M}, \ell^2_r) \)). The space \( L^{\text{cond}}_E(\mathcal{M}, \ell^2_\ell) \) (resp. \( L^{\text{cond}}_E(\mathcal{M}, \ell^2_r) \)) can be viewed as a closed subspace of the column (resp. row) subspace of \( L_E(\mathcal{M} \otimes \mathcal{B}(\ell^2_2)) \). We refer to [10,11,12] for more details on this.


For a finite noncommutative $L_E$-martingale $x = (x_n)_{n \geq 1}$ define (with $\mathcal{E}_0 = \mathcal{E}_1$)

$$\|x\|_{h^c_E(\mathcal{M})} = \|dx\|_{L^\text{cond}_E(\mathcal{M}, \ell^2_c)} \quad \text{and} \quad \|x\|_{h^r_E(\mathcal{M})} = \|dx\|_{L^\text{cond}_E(\mathcal{M}, \ell^2_r)}.$$

Let $h^c_E(\mathcal{M})$ and $h^r_E(\mathcal{M})$ be the corresponding completions. Then $h^c_E(\mathcal{M})$ and $h^r_E(\mathcal{M})$ are Banach spaces. We define the column and row conditioned square functions as follows. For any finite martingale $x = (x_n)_{n \geq 1}$ in $L_E(\mathcal{M})$, we set

$$s^c(x) = \left( \sum_{k \geq 1} \mathcal{E}_{k-1}(|dx_k|^2) \right)^{\frac{1}{2}} \quad \text{and} \quad s^r(x) = \left( \sum_{k \geq 1} \mathcal{E}_{k-1}(|dx^*_k|^2) \right)^{\frac{1}{2}}.$$

Then

$$\|x\|_{h^c_E(\mathcal{M})} = \|s^c(x)\|_{L_E(\mathcal{M})} \quad \text{and} \quad \|x\|_{h^r_E(\mathcal{M})} = \|s^r(x)\|_{L_E(\mathcal{M})}.$$
Let $x = (x_n)_{n \geq 0}$ be a finite $L_E$-martingale, we set

$$s^d(x) = \text{diag}(|dx_n|)$$

We note that

$$\|s^d(x)\|_{L_E(\mathcal{M} \otimes \mathcal{B}(\ell_2))} = \|dx_n\|_{L^d_E(\mathcal{M})}$$

Let $h^d_E(\mathcal{M})$ be the subspace of $L^d_E(\mathcal{M})$ consisting of all martingale difference sequences.
We define the Hardy space of noncommutative martingales and its conditioned version as follows. For \( 1 \leq p_E \leq q_E < 2 \),

\[
H_E(\mathcal{M}) = H_E^c(\mathcal{M}) + H_E^r(\mathcal{M}),
\]

equipped with the norm

\[
\|x\|_{H_E(\mathcal{M})} = \inf \left\{ \|y\|_{H_E^c(\mathcal{M})} + \|z\|_{H_E^r(\mathcal{M})} : x = y + z, \ y \in H_E^c(\mathcal{M}), \ z \in H_E^r(\mathcal{M}) \right\}
\]

and

\[
h_E(\mathcal{M}) = h_E^c(\mathcal{M}) + h_E^r(\mathcal{M}) + h_E^d(\mathcal{M}),
\]

equipped with the norm

\[
\|x\|_{h_E(\mathcal{M})} = \inf \left\{ \|y\|_{h_E^c(\mathcal{M})} + \|z\|_{h_E^r(\mathcal{M})} + \|w\|_{h_E^d(\mathcal{M})} : x = y + z + w, \ y \in h_E^c(\mathcal{M}), \ z \in h_E^r(\mathcal{M}), \ w \in h_E^d(\mathcal{M}) \right\}.
\]
For $2 \leq p_E \leq q_E < \infty$,

$$H_E(\mathcal{M}) = H^c_E(\mathcal{M}) \cap H^r_E(\mathcal{M}),$$

equipped with the norm

$$\|x\| = \max\{\|x\|_{H^c_E(\mathcal{M})}, \|x\|_{H^r_E(\mathcal{M})}\}.$$

and

$$h_E(\mathcal{M}) = h^c_E(\mathcal{M}) \cap h^r_E(\mathcal{M}) \cap h^d_E(\mathcal{M}),$$

equipped with the norm

$$\|x\|_{h_E(\mathcal{M})} = \max\{\|x\|_{h^c_E(\mathcal{M})}, \|x\|_{h^r_E(\mathcal{M})}, \|x\|_{h^d_E(\mathcal{M})}\}.$$
Using Lemma 6.4 in [10], Theorem 2.3 in [13] and Theorem 3.4 in [14] we obtain the following result.

**Proposition**

Let $E$ be a fully symmetric Banach space on $[0, \infty)$ with $1 < p_E \leq q_E < \infty$. Then we have

(i) $h^c_E(\mathcal{M}), h^r_E(\mathcal{M})$ are complemented in $L_E(\mathcal{M} \otimes \mathcal{B}(\ell_2(\mathbb{N}^2)))$.

(ii) $H^c_E(\mathcal{M}), H^r_E(\mathcal{M})$ are complemented in $L_E(\mathcal{M} \otimes \mathcal{B}(\ell_2))$.


By Theorem 5.6 in [15] and the above proposition, it follows that

**Theorem**

Let $E$ be a separable symmetric Banach space on $[0, \infty)$ with $1 < p_E \leq q_E < \infty$. Then we have

(i) $(h^c_E(\mathcal{M}))^* = h^c_{E \times}(\mathcal{M})$ with equivalent norms.

(ii) $(H^c_E(\mathcal{M}))^* = H^c_{E \times}(\mathcal{M})$ with equivalent norms.

Similarly, $(h^r_E(\mathcal{M}))^* = h^r_{E \times}(\mathcal{M})$ and $(H^r_E(\mathcal{M}))^* = H^r_{E \times}(\mathcal{M})$ with equivalent norms.

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Proposition

Let $E$ be a symmetric Banach space on $[0, \infty)$ with $2 < p_E \leq q_E < \infty$. Then we have

(i) $H^c_E(\mathcal{M})) = h^c_E(\mathcal{M}) \cap h^d_E(\mathcal{M})$ with equivalent norms.

(ii) $H^r_E(\mathcal{M}) = H^r_E(\mathcal{M}) \cap h^d_E(\mathcal{M})$ with equivalent norms.
Proposition

Let $E$ be a symmetric Banach space on $[0, \infty)$. If $E$ is separable or the dual of a separable space and satisfies $1 < p_E \leq q_E < 2$. Then we have

(i) $H^c_E(\mathcal{M})) = h^c_E(\mathcal{M}) + h^d_E(\mathcal{M})$ with equivalent norms.

(ii) $H^r_E(\mathcal{M}) = h^r_E(\mathcal{M}) + h^d_E(\mathcal{M})$ with equivalent norms.
Corollary

Let $E$ be a separable symmetric Banach space on $[0, \infty)$ with $1 < p_{E} \leq q_{E} < 2$. Then we have

$$L_{E}(\mathcal{M}) = h_{E}^{c}(\mathcal{M}) + h_{E}^{d}(\mathcal{M}) + h_{E}^{r}(\mathcal{M}).$$
Theorem

Let $E$ be a symmetric Banach space on $[0, \infty)$. If $E$ is separable or the dual of a separable space and satisfies $1 < p_E \leq q_E < 2$. Then we have

$$(h_E(\mathcal{M}))^* = h_{E \times}(\mathcal{M})$$

with equivalent norms.
We recall interpolation of noncommutative symmetric spaces. Let $E_1, E_2$ be fully symmetric spaces on $[0, \infty)$ and $0 < \theta < 1$. If $E$ is complex interpolation of $E_1$ and $E_2$, i.e. $E = (E_1, E_2)_\theta$. Then

$$L_E(M) = (L_{E_1}(M), L_{E_2}(M))_\theta.$$  \hfill (3.1)

Since $\{\text{diag}(a_n) : (a_n) \subset M\}$ is von Neumann subalgebra of $M \otimes B(\ell_2)$, we have that

$$L^d_E(M) = (L^d_{E_1}(M), L^d_{E_2}(M))_\theta.$$  \hfill (3.2)

For more details on interpolation of noncommutative symmetric spaces we refer to [14].
Proposition

Let \( E_j \) be a fully symmetric Banach spaces on \([0, \infty)\) with \(1 < p_{E_j} \leq q_{E_j} < 2\) \((j = 1, 2)\). If \(0 < \theta < 1\) and \(E = (E_1, E_2)_\theta\), then

(i)

\[
\begin{align*}
    h^c_E(\mathcal{M}) &= (h^c_{E_1}(\mathcal{M}), h^c_{E_2}(\mathcal{M}))_\theta, \\
    h^r_E(\mathcal{M}) &= (h^r_{E_1}(\mathcal{M}), h^r_{E_2}(\mathcal{M}))_\theta.
\end{align*}
\]

(ii)

\[
\begin{align*}
    H^c_E(\mathcal{M}) &= (H^c_{E_1}(\mathcal{M}), H^c_{E_2}(\mathcal{M}))_\theta, \\
    H^r_E(\mathcal{M}) &= (H^r_{E_1}(\mathcal{M}), H^r_{E_2}(\mathcal{M}))_\theta.
\end{align*}
\]
Let $E_j$ be a symmetric Banach space on $[0, \infty)$ with $1 < p_{E_j} \leq q_{E_j} < \infty$ ($j = 1, 2$) and $0 < \theta < 1$. Suppose that $E = (E_1, E_2)_{\theta}$, then the following holds with equivalent norms

$$(h_{E_1}^d(M), h_{E_2}^d(M))_{\theta} = h_E^d(M).$$
Interpolation

Theorem

Let $E_j$ be a symmetric Banach space on $[0, \infty)$ satisfying the Fatou property, and let $1 < p_{E_j} \leq q_{E_j} < 2$ ($j = 1, 2$). Suppose that either $E_1$ or $E_2$ has order continuous norm. If $0 < \theta < 1$ and $E = (E_1, E_2)_\theta$, then

(i) $h_E(M) = (h_{E_1}(M), h_{E_2}(M))_\theta$ holds with equivalent norms.

(ii) $H_E(M) = (H_{E_1}(M), H_{E_2}(M))_\theta$ holds with equivalent norms.
Let $E_j$ be a fully symmetric Banach space on $[0, \infty)$ with $2 < p_{E_j} \leq q_{E_j} < \infty$ ($j = 1, 2$). If $0 < \theta < 1$ and $E = (E_1, E_2)_\theta$, then

(i) $h_E(\mathcal{M}) = (h_{E_1}(\mathcal{M}), h_{E_2}(\mathcal{M}))_\theta$ holds with equivalent norms.

(ii) $H_E(\mathcal{M}) = (H_{E_1}(\mathcal{M}), H_{E_2}(\mathcal{M}))_\theta$ holds with equivalent norms.

Thanks for your attention!