Semigroup generators, H^∞-calculus, and BMO

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Based on joint work with Furguson, Hong, Junge, Simanek

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Quasi monotone property of semigroup of operators

A sample

Poisson integral operators $P_s, s > 0$:

$$P_s f = e^{-s\sqrt{\Delta}} f = \frac{1}{\pi}\frac{s}{s^2 + |\cdot|^2} * f(\cdot).$$

$$rP_s(f) \leq P_{rs}(f), \qquad s^k|\partial_s^k P_s f| \leq c_{k,r} P_{rs} f,$$

for $f \geq 0; 0 < r < 1, k = 0, 1, 2, \cdots$.

Question: Suppose L generates a semigroup of positive contractions on a Banach lattice X.

$$s^k|\partial_s^k e^{-sL^\alpha} f| \lesssim e^{-rsL^\alpha}|f| ??$$

for any $f \in X, 0 < r, \alpha < 1, k = 0, 1, 2, \cdots$?

Yes for $\alpha \leq \frac{1}{2}$. Because,

$$e^{-s\sqrt{L}} = \int_0^\infty e^{-\lambda L}\phi_s(\lambda) d\lambda$$

with

$$\phi_s(\lambda) = \frac{1}{2\sqrt{\pi}} s e^{-\frac{s^2}{4\lambda}} \lambda^{-\frac{3}{2}}, \quad s^k|\partial_s^k \phi_s(\lambda)| \lesssim \phi_{rs}(\lambda).$$

Subordinated Semigroups have Quasi Monotonicity

Yes for all $0 < \alpha < 1$!

Theorem

[Ferguson-M-Simanek 2017] Suppose L generates a semigroup of positive contractions on a Banach lattice X. Let

$$T_{s,\alpha} = e^{-sL^\alpha}.$$

For all $f \in X_+, 0 < r, \alpha < 1$, *and* $k \in \mathbb{N}$, *we have*

$$r^{\frac{1}{1-\alpha}} T_{s,\alpha} f \leq T_{rs,\alpha} f \tag{1}$$

$$|s^k \partial_s^k T_{s,\alpha}(f)| \leq (\frac{10k}{1-\alpha})^k T_{\alpha s,\alpha}(f). \tag{2}$$

$$s^{2k} |\nabla_L (\partial_s^k T_{s,\alpha} f)|^2 \leq \left(\frac{10k}{1-\alpha}\right)^{2k} T_{s,\alpha}(|\nabla_L f|^2) \tag{3}$$

Here

$$|\nabla_L f|^2 = L(\bar{f})f + \bar{f}L(f) - L(|f|^2).$$

Completely monotone functions

$f : \mathbb{R}_+ \to \mathbb{R}_+$ is *completely monotone* if

$$(-1)^k \partial_t^k f \geq 0, k = 0, 1, 2....$$

e.g. $f(x) = e^{-x}$.

Hausdorff-Bernstein:
$\varphi(x) = \int_0^\infty e^{-tx} d\mu(t)$ iff φ is completely monotone.
e.g. $0 < \alpha < 1, s > 0$,

$$\varphi(x) = e^{-tx^\alpha} = \int_0^\infty e^{-\lambda x} \phi_t(s) d\lambda.$$

Question: Is there a constant $c_{t,\alpha}$ such that

$$\varphi(x) = c_{\alpha,t} e^{-rx^\alpha} - e^{-x^\alpha}$$

is completely monotone for any $0 < r, \alpha < 1$?
Yes! Ferguson-M-Simanek 2017

Functional-calculus

For $\Delta = \partial_x^2$, $\Phi(\Delta)$ makes sense as Fourier multipliers,

$$\widehat{\Phi(\Delta)(f)}(\xi) = \Phi(|\xi|^2)\hat{f}(\xi).$$

For general unbounded operator L densely defined on a Banach space X, $\Phi(L)$ can be defined by
spectral theory + (complex) Fourier analysis.

H^∞-calculus L has bounded $H^\infty(\eta)$-calculus, if the map $\Phi(L)$ extends to a bounded operator on X and there is a constant C such that

$$\|\Phi(L)\| \leq C\|\Phi\|_{H^\infty(S_\eta)} \tag{4}$$

for any bounded analytic function $\Phi \in H^\infty(S_\eta)$. Here

$$S_\eta = \{z \in \mathbb{C} : |argz| < \eta\}.$$

Known result on H^∞-calculus

L: the infinitesimal generator of a semigroup of positive preserving contractions on $L^p, 1 < p < \infty$.

(Cowling, Duong, and Hieber & Prüss)
$\Rightarrow$ L has a bounded $H^\infty(\eta)$-caluclus, e.g. $\Phi(L)$ is bounded on L^p

for any $\Phi \in H^\infty(S_\eta)$ with $\eta > \frac{\pi}{2}$.
e.g. $L^{ti}, (1+L)^{-\alpha}$ and $exp(-iL^\alpha)$.....

Example Given $\varepsilon_k = \pm 1$, is there $m : \mathbb{Z} \to \mathbb{Z}$ such that $m(2^k) = \varepsilon_k$ such that

$$M_m : e^{ikx} \to m(\xi)e^{ikx}$$

extends to a bounded Fourier multiplier on $L^p(\mathbb{T})$, $1 < p < \infty$?
Yes. Δ has a bounded H^∞-caluclus on $L^p, 1 < p < \infty$
and $\forall \varepsilon_k = \pm 1$, there is bounded analytic function Φ on the right half plane so that $\Phi(4^k) = \varepsilon_k$. So

$$M_m = \Phi(\Delta) : e^{ikx} \to \Phi(|k|^2)e^{ikx}$$

is a solution.

Question: Bounded H^∞-calculus on BMO?

Semigroup-BMO

L: Infinitesimal generator of a weak*-continuous -semigroup of self adjoint, positive preserving contractions on $\mathcal{M}$.

$$\|f\|_{BMO(\sqrt{L})} = \sup_{t>0} \|e^{-t\sqrt{L}}|f - e^{-t\sqrt{L}}f|^2\|_\infty^{\frac{1}{2}}.$$

$$\|f\|_{bmo(\sqrt{L})} = \sup_{t>0} \|e^{-t\sqrt{L}}|f|^2 - |e^{-t\sqrt{L}}f|^2\|_\infty^{\frac{1}{2}}.$$

Theorem (Junge/Mei 2012)

$$BMO(\sqrt{L}) \simeq bmo(\sqrt{L}). \quad [BMO(\sqrt{L}), L^1(\mathcal{M})]_{\frac{1}{p}} = L^p(\mathcal{M}).$$

Remark Garsia, Varoupoulos, Duong-Yan, Liu, Yang,Yao,.......
Remark Martingale BMO theory, Junge-Perrin; Pisier-Xu, Musat, Randrianantoanina, Parcet, Bekjan-Chen, Hong-Mei, Jiao, Wu, Dirkson.....

Theorem

(Ferguson-M-Simanek 2017)
Suppose e^{-tL} is a weak continuous semigroup of positive contractions on L^∞. Suppose L satisfies Bakry-Émery's $\Gamma^2 \geq 0$ criterion. Then*

(i) $BMO(\sqrt{L}) \simeq BMO(L^\alpha) \simeq bmo(L^\alpha)$ for all $0 < \alpha < 1$.
(ii) L has a bounded $H^\infty(S_\eta)$ calculus on $BMO(\sqrt{L})$ for any $\eta > \frac{\pi}{2}$.

i.e. $\Phi(L)$ defines a bounded map on $BMO(\sqrt{L})$ if $\Phi \in H^\infty(S_\eta)$ for some $\eta > \frac{\pi}{2}$.

Remark The fact that $\sqrt{L}$ has a bounded $H^\infty(\eta)$ calculus on $BMO(\sqrt{L})$ for any $\eta > \frac{\pi}{2}$ is due to Mei-de la Salle, Junge-Mei.

Example:Ornstein-Uhlenbeck operator

Let

$$-L = \frac{\Delta}{2} - x \cdot \nabla$$

be the Ornstein-Uhlenbeck operator on $(\mathbb{R}^n, e^{-|x|^2}dx)$.
Then L has a bounded $H^\infty(\eta)$-calculus on $BMO(\sqrt{L})$ for any $\eta > \frac{\pi}{2}$.
Mauceri and Meda introduced the following BMO space

$$\|f\|_{BMO(MM)} = \sup_{r_B \leq \min\{1, \frac{1}{|c_B|}\}} (E_B^\mu |f - E_B^\mu f|^2)^{\frac{1}{2}},$$

with r_B, c_B the radius and the center of B, and $E_B^\mu = \frac{1}{\mu(B)} \int \cdot d\mu$.
One can verify that

$$\| \cdot \|_{BMO(MM)} \lesssim \| \cdot \|_{BMO(\sqrt{L})},$$

Therefore, L has bounded $H^\infty(S_\eta)$ calculus from $L^\infty(\mathbb{R}^n, e^{-|x|^2}dx)$ to Mauceri-Meda's BMO(MM) for any $\eta > \frac{\pi}{2}$.

Example: BMO Fourier Multipliers on Free groups

$\mathbb{F}_n$: Free group,
$|g|$: word length,
λ_g: left translation by g on $\ell_2(\mathbb{F}_n)$,
$L : \lambda_g \mapsto |g|\lambda_g$, Then

$$\Phi(L) : \lambda_g \mapsto \Phi(|g|)\lambda_g,$$

extends to a bounded map on $\mathrm{BMO}(\sqrt{L})$.

Corollary (Furguson-M-Simanek) Given $\varepsilon_k = \pm 1$, there is $m : \mathbb{R}_+ \to \mathbb{R}$ such that $m(2^k) = \varepsilon_k$ such that

$$M_m : \lambda_h \to m(|h|)\lambda_h$$

is bounded on $\mathrm{BMO}(\sqrt{L})$.

Remark L^p result is due to Junge-Le Merdy-Xu 2006.

Bakry-Émery's $\Gamma^2 \geq 0$ criterion

e^{-tL} is unital positive preserving $\Rightarrow$

$$e^{-sL}|e^{-(t-s)L}f|^2$$

increases in s on $(0,t)$.
We say L satisfies the $\Gamma^2 \geq 0$ criterion if

$$e^{-sL}|e^{-(t-s)L}f|^2$$

is convex in s on $(0,t)$, i.e.

$$2e^{-\frac{t}{2}L}|e^{-\frac{t}{2}L}f|^2 \leq e^{-tL}|f|^2 + |e^{-tL}f|^2.$$

Proposition (Furguson-M-Simanek) Suppose L satisfies Bakry-Émery's $\Gamma^2 \geq 0$ criterion, then L^α satisfies it too for all $0 < \alpha < 1$.

Why passing to L^α helps?

Recall that L has bounded $H^\infty(\eta)$ calculus
$\Rightarrow$ $(\Leftarrow)$ $\|L^{is}\|_{X\to X} \leq c\exp(\eta|s|)$.
Using

$$L^{i\alpha s} = \Gamma(1-is)^{-1}\int_0^\infty t^{-is} L^\alpha e^{-tL^\alpha}\,dt.$$

and the quasi monotonicity of e^{-tL^α}, we obtain

$$\|L^{i\alpha s} f\|_{BMO(L^\alpha)} \leq \frac{c}{(1-\alpha)^2(1+|s|)^{\frac{1}{2}}}\exp\left(\frac{\pi|s|}{2\alpha}\right)\|f\|_{BMO(L^\alpha)}.$$

Choosing $\alpha = \frac{|s|}{|s|+1}$ for s large and using the equivalence $BMO(L^\alpha) \simeq BMO(\sqrt{L})$, we get for $\eta > \frac{\pi}{2}$,

$$\begin{aligned}\|L^{is} f\|_{BMO(\sqrt{L})} &\leq cP(s)\exp\left(\frac{\pi|s|}{2}\right)\|f\|_{BMO(\sqrt{L})}\\ &\leq c\exp(\eta|s|)\|f\|_{BMO(\sqrt{L})}\end{aligned}$$

if $\eta > \frac{\pi}{2}$.