Singular integrals and PDEs in quantum Euclidean spaces

Javier Parcet

Probabilistic Approach to Harmonic Analysis

Wuhan – May 19-21, 2017

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Based on joint work with
A. González-Pérez and M. Junge

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Heisenberg-Weyl algebra

POSITION/MOMENTUM IN QUANTUM MECHANICS: $(\mathbf{x}, \mathbf{p}) = (x, i\hbar\partial_x)$

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HW algebra – Unbounded generators

Given $f \in L_2(\mathbb{R}^d)$, let us set

$$\left. \begin{array}{rcl} A_j f(x) & = & x_j f(x) \\ B_j f(x) & = & i\hbar\partial_{x_j} f(x) \end{array} \right\} \Rightarrow \ [A_j, B_j] = -i\hbar.$$

The Heisenberg-Weyl algebra $\mathcal{R}_{\mathrm{hw}}$ is *formally* generated by A_j and B_j.

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HW algebra – Unitary representation

Letting $u_j(s) = \exp(2\pi i s A_j)$ and $v_j(s) = \exp(2\pi i s B_j)$ we find

$$u_j(s), v_j(s) \in \mathcal{B}(L_2(\mathbb{R}^d)),$$

$$u_j(s_1)v_j(s_2) = e^{2\pi i\hbar s_1 s_2} v_j(s_2)u_j(s_1).$$

The Heisenberg-Weyl algebra $\mathcal{R}_{\mathrm{hw}}$ is defined as $\langle u_j(s_1), v_k(s_2)\rangle'' \subset \mathcal{B}(L_2(\mathbb{R}^d))$.

Quantum Euclidean spaces

Given $\Theta \in M_n(\mathbb{R})$ anti-symmetric, let

$$\mathcal{R}_\Theta = \Big\langle u_1(s), u_2(s), \ldots, u_n(s) : s \in \mathbb{R} \Big\rangle''$$

in $\mathcal{B}(L_2(\mathbb{R}^n))$, subject to $u_j(s)$ unitaries with

$$u_j(s_1 + s_2) = u_j(s_1) u_j(s_2),$$
$$u_j(s_1) u_k(s_2) = e^{2\pi i \Theta_{jk} s_1 s_2} u_k(s_2) u_j(s_1).$$

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Basic definitions

- **Characters** $\lambda_\Theta(\xi) = u_1(\xi_1)u_2(\xi_2)\cdots u_n(\xi_n)$.
- **Fourier transform** $\lambda_\Theta(f) = \displaystyle\int_{\mathbb{R}^n} f(\xi)\lambda_\Theta(\xi)d\xi$.
- L_p **spaces** $\tau_\Theta(\lambda_\Theta(f)) = f(0)$ and $\|\varphi\|_p^p = \tau_\Theta|\varphi|^p$.
- **Quantum Euclidean variables** $x_{\Theta,j} = \dfrac{1}{2\pi i}\dfrac{d}{ds}\Big|_{s=0}(u_j(s))$

Where do we find these spaces?

Algebraically, $\mathcal{R}_\Theta$ is a type I algebra
Indispensable in a great variety of scenarios...

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Quantum tori

$$\mathcal{A}_\Theta = \lambda_\Theta(\mathbb{Z}^n)'' \subset \mathcal{R}_\Theta$$

the $\mathbb{Z}^n$-periodic subalgebra of $\mathcal{R}_\Theta$. Quantum 2-tori = Model of NC geometry.

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Too many names for $\mathcal{R}_\Theta \rightsquigarrow$ **QUANTUM EUCLIDEAN SPACES**

Singular integrals and PDEs – Why?

Harmonic analysis and PDEs over Riemannian manifolds are paramount for the solution of important problems in differential geometry and theoretical physics

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Lack of elementary geometric methods for PDEs
ΨDOs require (a priori) less geometric background

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Calderón-Zygmund theory in $\mathcal{R}_\Theta$ $\rightsquigarrow$ Noncommutative singular kernels!

Calderón-Zygmund theory – What is it?

Given a Riemannian manifold $(\mathrm{X}, \mathrm{d}, \mu)$

$$T_k f(x) = \int_{\mathrm{X}} \underbrace{k(x,y)}_{\substack{\text{distributional kernel} \\ \text{singular at diagonal } x=y}} f(y)\, d\mu(y) \quad \text{for} \quad x \notin \mathrm{supp} f.$$

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The paradigm of singular integral theory is the Hilbert transform in $\mathbb{R}$, paramount to study the convergence of Fourier series. The challenge in higher dimensions required new real variable methods which culminated in...

Calderón-Zygmund theorem – [Acta Math, 1952]

Assume T_k satisfies:

i) **Cancellation** $\big\| T_k : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n) \big\| \le \mathrm{A}_1$.

ii) **Kernel smoothness** $\big|\nabla_x k(x,y)\big| + \big|\nabla_y k(x,y)\big| \le \dfrac{\mathrm{A}_2}{|x-y|^{n+1}}$.

Then, $T_k : L_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n)$ defines a bounded map for $1 < p < \infty$.

The same holds in Riemannian manifolds with nonnegative Ricci curv [Bakry, 1987].

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PROBLEM. Calderón-Zygmund conditions in $\mathrm{X} = \mathcal{R}_\Theta$? We have no points in $\mathcal{R}_\Theta$!

—Junge-Mei-Parcet 2010–2015—

NC Singular Integrals and Fourier Multipliers

Calderón-Zygmund theory – Quantum kernels

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NC Singular Integrals and Fourier Multipliers

A) Where does the **kernel** live?

B) If there are no points, what is the **diagonal**?

C) If we have no metric, how to quantify **singularities**?

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KEY NOVELTY: The kernel must be k affiliated to $\mathcal{R}_\Theta \bar{\otimes} \mathcal{R}_\Theta^{\mathrm{op}}$ – This is **crucial**!!

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What is a quantum singular kernel?

The map

$$\pi_\Theta : L_\infty(\mathbb{R}^n) \ni \exp_\xi \to \lambda_\Theta(\xi) \otimes \lambda_\Theta(\xi)^* \in \mathcal{R}_\Theta \bar{\otimes} \mathcal{R}_\Theta^{\mathrm{op}}$$

is a quantum form of $\pi f(x,y) = f(x-y)$, where the op-structure is essential.

A) Kernel: $k \in \mathcal{R}_\Theta \bar{\otimes} \mathcal{R}_\Theta^{\mathrm{op}}$.

B) Kernel diagonal truncation: $\pi_\Theta(1-\phi_\varepsilon) \bullet k \bullet \pi_\Theta(1-\phi_\varepsilon)$.

C) Quantum metric and kernel singularities:

$$\mathrm{d}_\Theta^\alpha \bullet k \bullet \mathrm{d}_\Theta^\beta \lesssim 1 \quad \text{for} \quad \mathrm{d}_\Theta = \pi_\Theta(\mathrm{dist}) \quad \text{and} \quad \alpha + \beta = n.$$

Calderón-Zygmund theory – Main result

Quantum derivatives: $\partial_\Theta^j(\lambda_\Theta(\xi)) = 2\pi i \xi_j \lambda_\Theta(\xi) \rightsquigarrow \nabla_\Theta = \sum_j s_j \otimes \partial_\Theta^j.$

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Theorem A [González-Pérez/Junge/Parcet, Preprint'17]

Let $T_k \in \mathcal{L}(\mathcal{S}_\Theta, \mathcal{S}'_\Theta)$ and assume:

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for $(\alpha, \beta) = (n+1, 0)$, $(\alpha, \beta) = (0, n+1)$ and $(\alpha, \beta) = \frac{1}{2}(n+1, n+1)$.

Then, $T_k : L_p(\mathcal{R}_\Theta) \to L_p(\mathcal{R}_\Theta)$ is a completely bounded map for all $1 < p < \infty$.

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Remarks.

- **BMO endpoint** via quantum heat process.
- L_p **interpolation:** NC martingales and Markov dilations.
- **Nonconvolution case:** Highly technical, crucial for ΨDOs.
- **Existence of principal values:** Ok up to pointwise multipliers.

Pseudodifferential calculus – What is it?

- **Fourier representation**

$$Lf(x) := \sum_\alpha a_\alpha(x) \partial_x^\alpha f(x) = \int_{\mathbb{R}^n} a(x,\xi) \widehat{f}(\xi) e^{2\pi i \langle x, \xi \rangle} d\xi =: \Psi_a f(x).$$

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- **Parametrices and error terms**

$$\Pi \circ \Psi_a = id - \mathcal{E} \ \rightsquigarrow\ \Psi_a f = g \ \Leftrightarrow\ f = \Pi g + \mathcal{E}f.$$

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- **Smoothness test for Sobolev p-estimates**

$$\big|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)\big| \ \le \ C_{\alpha\beta}\big(1+|\xi|\big)^{m-\rho|\alpha|+\delta|\beta|}.$$

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Euclidean theory
Since the mid 1960's by Kohn, Nirenberg, Hörmander, Feffermann, Stein...

Noncommutative geometry
Connes 1980: $\mathcal{C}^\infty$-theory + Atiyah-Singer index theorem for NC dynamical systems

Pseudodifferential calculus – Quantum ΨDOs

The symbol $a(x,\xi)$ becomes $a : \mathbb{R}^n \to \mathcal{R}_\Theta$ and

$$\Psi_a(\lambda_\Theta(f)) = \int_{\mathbb{R}^n} a(\xi) f(\xi) \lambda_\Theta(\xi)\, d\xi \quad \rightsquigarrow \quad \mathbf{kernel} = \int_{\mathbb{R}^n} (a(\xi) \otimes \mathbf{1}) \pi_\Theta(\exp_\xi)\, d\xi$$

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Two possible quantum forms of Hörmander classes

- **Hörmander class** $S^m_{\rho,\delta}(\mathcal{R}_\Theta)$

$$\big| \partial_\Theta^\beta \partial_\xi^\alpha a(\xi) \big| \le C_{\alpha\beta} \big(1 + |\xi| \big)^{m - \rho|\alpha| + \delta|\beta|}.$$

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$$\big|\partial_\Theta^\beta \partial_{\Theta,\xi}^{\alpha_1} \partial_\xi^{\alpha_2} a(\xi)\big| \le C_{\alpha_1\alpha_2\beta} \big(1+|\xi|\big)^{m-\rho|\alpha_1+\alpha_2|+\delta|\beta|}.$$

Here $\partial_{\Theta,\xi}$ is a **Θ-deformation of ∂_ξ by ∂_Θ's**:

$$\begin{aligned} \partial^j_{\Theta,\xi} a(\xi) &= \partial_\xi^j a(\xi) + \frac{1}{2\pi i} \sum_{k=1}^n \Theta_{jk}\, \partial_\Theta^k a(\xi) \\ &= \lambda_\Theta(\xi)^* \partial_\xi^j \big\{ \lambda_\Theta(\xi) a(\xi) \lambda_\Theta(\xi)^* \big\} \lambda_\Theta(\xi). \end{aligned}$$

Pseudodifferential calculus – Main results

In addition to Connes $\mathcal{C}^\infty$-theory, we develop:

i) The L_2-theory for the Hörmander classes $S^m_{\rho,\delta}(\mathcal{R}_\Theta)$

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The reason for $\Sigma^m_{\rho,\delta}(\mathcal{R}_\Theta) \subset S^m_{\rho,\delta}(\mathcal{R}_\Theta)$ comes from modularity in NC CZ theory!

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Theorem B [González-Pérez/Junge/Parcet, Preprint'17]

Let $a : \mathbb{R}^n \to \mathcal{R}_\Theta$ and $1 < p < \infty$:

i) If $a \in S^0_{\rho,\rho}(\mathcal{R}_\Theta)$ with $0 \le \rho < 1$, $\Psi_a : L_2(\mathcal{R}_\Theta) \to L_2(\mathcal{R}_\Theta)$.

ii) If $a \in S^0_{1,1}(\mathcal{R}_\Theta) \cap S^0_{1,1}(\mathcal{R}_\Theta)^*$, then $\Psi_a : L_2(\mathcal{R}_\Theta) \to L_2(\mathcal{R}_\Theta)$.

iii) If $a \in \Sigma^0_{1,1}(\mathcal{R}_\Theta) \cap \Sigma^0_{1,1}(\mathcal{R}_\Theta)^*$, then $\Psi_a : L_p(\mathcal{R}_\Theta) \to L_p(\mathcal{R}_\Theta)$.

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Using $S^m_{\rho,\delta}(\mathcal{R}_\Theta) \subset S^m_{\rho,\rho}(\mathcal{R}_\Theta) \cap S^m_{\delta,\delta}(\mathcal{R}_\Theta)$ (same for Σ) $\rightsquigarrow$ Core of the L_p-theory...

- Thm B i) = Quantum Calderón-Vaillacourt theorem [PNAS, 1972].
- Thm B ii) = Quantum Bourdaud condition $\sim T(1)$ theorem for ΨDOs.
- Thm B iii) = Link to Thm A requires the quantum-classical derivative $\partial_{\Theta,\xi}$.
- Sobolev p-estimates for symbols of degree m follow trivially from our results.
- L_p-estimates for $\rho < 1$ hold below Fefferman's critical index $= -(1-\rho)n/2$.
- Thm B and applications hold for quantum tori $\mathcal{A}_\Theta$ as well – Connes setting.

L_p regularity for elliptic PDEs

Theorem C [González-Pérez/Junge/Parcet, Preprint'17]

Given $1 < p < \infty$ and $r, s, m \in \mathbb{R}$...

Let $u \in \mathrm{W}_{p,r}(\mathcal{R}_\Theta)$ be a solution of the elliptic PDE

$$\Psi_a(u) = \varphi \quad \text{with} \quad \begin{cases} \varphi \in \mathrm{W}_{p,s}(\mathcal{R}_\Theta), \\ a \in \Sigma^m_{1,\delta}(\mathcal{R}_\Theta) \text{ elliptic.} \end{cases}$$

Then, $u \in \mathrm{W}_{p,s+m}(\mathcal{R}_\Theta)$ and the estimate below holds

$$\|u\|_{\mathrm{W}_{p,s+m}(\mathcal{R}_\Theta)} \lesssim \|u\|_{\mathrm{W}_{p,r}(\mathcal{R}_\Theta)} + \|\varphi\|_{\mathrm{W}_{p,s}(\mathcal{R}_\Theta)}.$$

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Theorem C generalizes for $p = 2$...

- Elliptic regularity with data in $S^m_{1,\delta}(\mathcal{R}_\Theta)$.
- Hypoellitic regularity with data in $S^m_{\rho,\delta}(\mathcal{R}_\Theta)$ for $\rho < 1$.
- We can quantify the regularity in terms of ρ, δ and n as in $\mathbb{R}^n$.

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Thank you!