Characterization of BMO and CMO via Commutators in Bessel Setting

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Contents

1. Abstract

In this talk, I'll present some recent results on the characterization of BMO space or CMO space in Bessel setting via the boundedness or compactness for the commutators of Riesz transform associated to Bessel operators in L^p-spaces and Morrey spaces. These are jointed works with Duong, Li, Mao and Yang.

2. Backgrounds

For all suitable functions f on $\mathbb{R}_+ := (0, \infty)$, consider the Bessel operator as follows:

$$\Delta_\lambda f(x) := -\frac{d^2}{dx^2} f(x) - \frac{2\lambda}{x}\frac{d}{dx} f(x).$$

Let $\{W_t^{[\lambda]}\}_{t>0}$, $\{P_t^{[\lambda]}\}_{t>0}$, R_{Δ_λ} be the heat and Poisson semigroup operators, Riesz transform associated with Δ_λ defined by setting, respectively, for all $f \in \bigcup_{1\le p\le\infty} L^p(\mathbb{R}_+, dm_\lambda)$ and $x \in \mathbb{R}_+$,

$$W_t^{[\lambda]}f(x) := e^{-t\Delta_\lambda}f(x) = \int_0^\infty W_t^{[\lambda]}(x,y)f(y)\,dm_\lambda(y),$$

$$P_t^{[\lambda]}f(x) := e^{-t\sqrt{\Delta_\lambda}}f(x) = \int_0^\infty P_t^{[\lambda]}(x,y)f(y)\,dm_\lambda(y)\,,$$

$$\begin{aligned} R_{\Delta_\lambda}f(x) : &= \partial_x(\Delta_\lambda)^{-1/2}f(x) \\ &= p.v. \int_{\mathbb{R}_+} R_{\Delta_\lambda}(x,y)f(y)dm_\lambda(y), \end{aligned}$$

where $dm_\lambda(y) = y^{2\lambda}dy$, $W_t^{[\lambda]}(x,y)$, $P_t^{[\lambda]}(x,y)$, $R_{\Delta_\lambda}(x,y)$ is the heat and Poisson kernel, the Riesz transform kernel, respectively.

2.1. Boundedness of singular integrals in Bessel setting

• Muckenhoupt and Stein [Trans. Amer. Math. Soc, 1965]

Developed a theory associated to Δ_λ which is parallel to the classical one associated to the Laplace operator Δ. For example, L^p bounds for H_λ (the Hankel transform), I_α (fractional integral) were given, where

$$\Delta_\lambda f = H_\lambda(x^2 H_\lambda(f)).$$

- Anderson-Muckenhoupt [Studia Math. 1981/1982]

H_λ, $\lambda \neq -1/2$, is bounded on $L^p(\omega)$ if and only if $\omega \in A_{p,\lambda}$, $1 < p < \infty$; moreover, weak type $(1,1)$.

- Villani [Illinois J. Math. 2008]

$$\|R_{\Delta_\lambda}(f)\|_{L^p(\mathbb{R}_+, dm_\lambda)} \approx \|f\|_{L^p(\mathbb{R}_+, dm_\lambda)},$$
$$\|R_{\Delta_\lambda}(f)\|_{L^p(\mathbb{R}_+, x^\alpha dm_\lambda)} \approx \|f\|_{L^p(\mathbb{R}_+, x^\alpha dm_\lambda)},$$
$$1 < p < \infty, \quad \alpha \in (-2\lambda, 2\lambda(p-1)).$$

• Betancor-Farina etc [Glasg. Math. J. 2009]:

$$g_\lambda^{(k,m)}f(x) := \Big(\int_0^\infty \Big| t^{k+m} \frac{\partial^{m+k}}{\partial t^k \partial x^m} P_t^{[\lambda]}(f)(x) \Big|^2 \frac{dt}{t} \Big)^{1/2}$$

(Higher order Littlewood-Paley g-function):

$$\star \quad L^p(\mathbb{R}_+, \omega(x)dm_\lambda) \to L^p(\mathbb{R}_+, \omega(x)dm_\lambda)$$

for $\omega \in A_p(\mathbb{R}_+)$, $1 < p < \infty$;

$$\star \quad L^1(\mathbb{R}_+, \omega(x)dm_\lambda) \to L^{1,\infty}(\mathbb{R}_+, \omega(x)dm_\lambda)$$

for $\omega \in A_1(\mathbb{R}_+)$;

$$\star \quad L_0^\infty(\mathbb{R}_+, dm_\lambda) \to BMO(\mathbb{R}_+, dm_\lambda);$$

$$\star \quad H^1(\mathbb{R}_+, dm_\lambda) \to L^1(\mathbb{R}_+, dm_\lambda).$$

• Betancor-Ruiz etc[J. Math. Anal. Appl. 2010]:

Let BMO_+ denote the space that consists of all those $f \in L^1_{loc}([0,\infty))$ such that the odd extension f_0 of f to $\mathbb{R}$ is in $BMO(\mathbb{R})$, $\mathcal{N}$ denote the operators M_0, W^*_λ, P^*_λ, R_{Δ_λ}, $g_{h,\lambda}$, $g_{P,\lambda}$, where M_0 is the HL-maximal operator on $(0,\infty)$, and

$$g_{h,\lambda}f(x) := \Big(\int_0^\infty \Big|t\frac{\partial}{\partial t}W_t^{[\lambda]}(f)(x)\Big|^2\frac{dt}{t}\Big)^{1/2},$$

$$g_{P,\lambda}f(x) := \Big(\int_0^\infty \Big|t\frac{\partial}{\partial t}P_t^{[\lambda]}(f)(x)\Big|^2\frac{dt}{t}\Big)^{1/2}.$$

Then for $\lambda > 0$,

$$\|\mathcal{N}f\|_{BMO_+} \le C\|f\|_{BMO_+}, \quad f \in BMO_+.$$

• Betancor-Harboure etc[Studia Math. 2010]:

Let $\lambda > -1/2$, $1 \le p < \infty$, $\delta \in \mathbb{R}$, the measure space be $(\mathbb{R}_+, x^\delta dx)$. Then

$*$ W_λ^*, P_λ^*: Strong type (∞, ∞);

$$strong\,type\,(p,p) \Leftrightarrow p > 1,\ -1 < \delta < (2\lambda+1)p - 1;$$

$$weak\,type\,(p,p) \Leftrightarrow -1 < \delta < (2\lambda+1)p-1, \text{ or } \delta = 2\lambda;$$

$$restrict\,weak\,type\,(p,p) \Leftrightarrow -1 < \delta \le (2\lambda+1)p - 1.$$

$*$ R_{Δ_λ}:

$$strong\,type\,(p,p) \Leftrightarrow p > 1,\ -1-p < \delta < (2\lambda+1)p-1;$$

$$weak\,type\,(p,p) \Leftrightarrow -1-p < \delta < (2\lambda+1)p-1, \text{ or } \delta \in \{2, 2\lambda\};$$

$$restrict\,weak\,type\,(p,p) \Leftrightarrow -1-p \le \delta \le (2\lambda+1)p-1.$$

$*$ $R^*_{\Delta_\lambda}$ (the adjoint of R_{Δ_λ}):

strong type $(p,p) \Leftrightarrow p > 1, -1 < \delta < (2\lambda+1)p - 1$;

weak type $(p,p) \Leftrightarrow -1 < \delta < (2\lambda+1)p-1$, or $\delta = 2\lambda+1$

restrict weak type $(p,p) \Leftrightarrow -1 \le \delta \le (2\lambda+1)p - 1$.

Moreover,

$$R^*_{\Delta_\lambda} R_{\Delta_\lambda} f = R_{\Delta_\lambda} R^*_{\Delta_\lambda} f = f, f \in L^p(\mathbb{R}_+, x^\delta dx)$$

for $p > 1$ and $-1 < \delta < (2\lambda+1)p - 1$.

$*$ $g_{h,\lambda}$:

strong type $(p,p) \Leftrightarrow p > 1, -1 < \delta < (2\lambda+1)p - 1$;

weak type $(p,p) \Leftrightarrow -1 < \delta < (2\lambda+1)p-1$, or $\delta = 2\lambda$;

restrict weak type $(p,p) \Leftrightarrow -1 \le \delta \le (2\lambda+1)p - 1$.

2.2. Characterization of Hardy spaces in Bessel setting

- Betancor-Dziubanski etc [J. Anal. Math. 2009]:

$$H^1_{atom}(dm_\lambda) \Leftrightarrow H^1_{\max}(\Delta_\lambda) \Leftrightarrow H^1_{Riesz}(\Delta_\lambda).$$

- Yang Da. and Yang Do. [Anal. Appl. 2011]

$$H^p_{atom}(dm_\lambda) \Leftrightarrow H^p_{\max-3}(\Delta_\lambda) \Leftrightarrow H^1_{Riesz}(\Delta_\lambda)$$
$$\Leftrightarrow H^p_{LP-g}(\Delta_\lambda) \Leftrightarrow H^p_{LS}(\Delta_\lambda),$$

where $\lambda > 0$, $(2\lambda+1)/(2\lambda+2) < p \leq 1$.

2.3. Boundedness of commutators and characterization of BMO space in Bessel setting

- Duong-Li-Wick-Yang [Indiana Univ. Math. J. 2015]:

$[b, R_{\Delta_\lambda}]$ defined by $[b, R_{\Delta_\lambda}](f) = bR_{\Delta_\lambda}(f) - R_{\Delta_\lambda}(bf)$:

$$L^p(\mathbb{R}_+, dm_\lambda) \to L^p(\mathbb{R}_+, dm_\lambda) \Longleftrightarrow$$

$$b \in BMO(\mathbb{R}_+, dm_\lambda), \quad 1 < p < \infty,\ \lambda > 0,$$

where

$$\sup_{x,r\in\mathbb{R}_+} \frac{1}{m_\lambda(I(x,r))} \int_{I(x,r)} \left| b(y) - b_{I(x,r),\lambda} \right| dm_\lambda(y) < \infty,$$

$$b_{I(x,r)} = \frac{1}{m_\lambda(I(x,r))} \int_{I(x,r)} b(y) dm_\lambda(y).$$

• Ding [Northeast Math. J. 1997]:

Let $[b,T]$ be the commutator of Calderon-Zygmund singular integral operator T on $\mathbb{R}^n$, $M^{p,\beta}(\mathbb{R}^n)$ be the Morrey spaces. Then for $1<p<\infty$, $-n/p\le\beta<0$,

$$[b,T]:M^{p,\beta}(\mathbb{R}^n)\to M^{p,\beta}(\mathbb{R}^n)\Leftrightarrow b\in BMO(\mathbb{R}^n).$$

Question 1. Does the following result for the commutators of Riesz transforms in Bessel setting hold?

$$[b,R_{\Delta_\lambda}]:M^{p,\beta}(\mathbb{R}_+,dm_\lambda)\to M^{p,\beta}(\mathbb{R}_+,dm_\lambda)$$
$$\Leftrightarrow b\in BMO(\mathbb{R}_+,dm_\lambda).$$

2.4. Compactness of commutators and characterization of CMO space

• Uchiyama [Tohoku Math. J. 1978]:

$[b, T_\Omega]$ is compactness on $L^p(\mathbb{R}^n)$

$$\Longleftrightarrow b \in CMO(\mathbb{R}^n), \quad 1 < p < \infty,$$

where T_Ω is the convolution singular integral with homogeneous kernel $\Omega \in Lip_1(S^{n-1})$, $CMO(\mathbb{R}^n)$ is the closure in $BMO(\mathbb{R}^n)$ of the space of C_c^∞ functions.

• Beatrous-Li [J. Funct. Anal. 1993]; Krantz-Li [J. Math. Anal. Appl. 2001]:

The L^p-compactness characterization of $[b, T_\Omega]$ over some spaces of homogeneous type and applying to one of Hankel operator on holomorphic Hardy space.

• Chen- Ding [Sci. China Math. 2010]:

Improved and extended the result of Uchiyama [Tohoku Math. J. 1978] to the commutator of parabolic singular integral with kernel Ω satisfying certain L^q-Dini's condition.

• Chen-Ding-Wang [Canad. J. Math. 2012]:

Extended the result of Uchiyama [Tohoku Math. J. 1978] to the Morrey spaces.

• Chen-Hu [Canad. Math. Bull. 2015]; Guo-Hu [Sci. China Math. 2015]; Chen-Chen-Hu [J. Math. Anal. Appl. 2015]:

The L^p-compactness of $[b, T_\Omega]$ with rough kernel and rough variable kernel Ω.

• Wang[Chin. Ann. Math. 1987]; Chen-Ding-Wang [Potential Anal. 2009]:

The $L^p - L^q$ and $M^{p,\beta} - M^{q,\beta}$ compactness characterization of $CMO(\mathbb{R}^n)$ for the commutators of fractional integrals.

• Chen-Ding [Kodai Math. J. 2009], Chen-Ding-Wang [Taiwanese J. Math. 2011], Mao-Sawano-Wu [Taiwnaese J. Math. 2015]:

The compactness of the commutators of Littlewood-Paley operators, Marcinkiewicz integrals in L^p-spaces and $M^{p,\beta}$-spaces.

• Ding-Mei [Studia Math. 2017]:

The L^p compactness for Calderon type commutators.

• Benyi-Torres[Proc. Amer. Math. Soc. 2013], Bu-Chen [arXiv:1411.1905v1], Hu [Taiwanese J. Math. 2014], Zhou-Li[J. Funct. Spaces 2014],Ding-Mei [Potential Anal. 2015],Bu-Guo-Huang [Banach J. Math. Anal. 2015], Mao-Sun-Wu[Acta Math. Sin. 2016], Mao-Wu [Bull. Korean Math. Soc. 2016], Ding-Mei-Xue[Adv. Lect. Math. 2016], Hu [Chin. Ann. Math. Ser. B 2017] etc.:

The compactness of the commutators and maximal commutators of multilinear operators (singular integrals, fractional integrals and Fourier multipliers) in L^p-spaces and Morrey spaces and the corresponding weighted versions.

• Li-Mo-Zhang [Math. Nachr. 2015]:

The compactness of the commutators for the singular integrals associated to Schrödinger operators in L^p-spaces.

• Question 2. Are the following results for the commutators of Riesz transforms in Bessel setting true?

(i) $[b, R_{\Delta_\lambda}]$ is compactness on $L^p(\mathbb{R}_+, x^{2\lambda}dx)$,

$$\Longleftrightarrow b \in CMO(\mathbb{R}_+, x^{2\lambda}dx), \quad 1 < p < \infty.$$

(ii) $[b, R_{\Delta_\lambda}]$ is compactness on $M^{p,\beta}(\mathbb{R}_+, x^{2\lambda}dx)$,

$$\Longleftrightarrow b \in CMO(\mathbb{R}_+, x^{2\lambda}dx), \quad 1 < p < \infty.$$

Here $CMO(\mathbb{R}_+, x^{2\lambda}dx)$ is the BMO-closure of the $C_c^\infty(\mathbb{R}_+)$ in Bessel setting.

3. Our main results in Bessel setting

Theorem 1 [DLMWY-JAM, 2019] Let $b \in \mathrm{BMO}(\mathbb{R}_+, dm_\lambda)$. Then $b \in \mathrm{CMO}(\mathbb{R}_+, dm_\lambda)$ if and only if the Riesz transform commutator $[b, R_{\Delta_\lambda}]$ is compact on $L^p(\mathbb{R}_+, dm_\lambda)$ for any $p \in (1, \infty)$.

Theorem 2 [MWY] Let $1 < p < \infty$, $0 < \beta < 1$ and $b \in \bigcup_{p>1} L^p_{\rm loc}(\mathbb{R}_+, dm_\lambda)$. Then

(i) If $b \in {\rm BMO}(\mathbb{R}_+, dm_\lambda)$, then the commutator $[b, R_{\Delta_\lambda}]$ is bounded on $M^{p,\beta}(\mathbb{R}_+, dm_\lambda)$ with the operator norm

$$\left\|[b, R_{\Delta_\lambda}]\right\|_{M^{p,\beta}(\mathbb{R}_+, dm_\lambda)\to M^{p,\beta}(\mathbb{R}_+, dm_\lambda)} \lesssim \|b\|_{{\rm BMO}(\mathbb{R}_+, dm_\lambda)}$$

(ii) If the commutator $[b, R_{\Delta_\lambda}]$ is bounded on $M^{p,\beta}(\mathbb{R}_+, dm_\lambda)$, then $b \in {\rm BMO}(\mathbb{R}_+, dm_\lambda)$ and

$$\|b\|_{{\rm BMO}(\mathbb{R}_+, dm_\lambda)} \lesssim \left\|[b, R_{\Delta_\lambda}]\right\|_{M^{p,\beta}(\mathbb{R}_+, dm_\lambda)\to M^{p,\beta}(\mathbb{R}_+, dm_\lambda)}$$

Theorem 3 [MWY] Let $0 < \beta < 1$, $1 < p < \infty$ and $b \in {\rm BMO}(\mathbb{R}_+, dm_\lambda)$. Then $b \in {\rm CMO}(\mathbb{R}_+, dm_\lambda)$ if and only if the Riesz transform commutator $[b, R_{\Delta_\lambda}]$ is compact on $M^{p,\beta}(\mathbb{R}_+, dm_\lambda)$.

Here $M^{p,\beta}(\mathbb{R}_+, dm_\lambda)$ denotes the Morrey space in Bessel setting for $1 < p < \infty$ and $0 < \beta < 1$, which is defined by

$$M^{p,\beta}(\mathbb{R}_+, dm_\lambda) := \{f \in L^p_{\mathrm{loc}}(\mathbb{R}_+, dm_\lambda) : \|f\|_{M^{p,\beta}(\mathbb{R}_+, dm_\lambda)} < \infty\},$$

where

$$\|f\|_{M^{p,\beta}(\mathbb{R}_+, dm_\lambda)} := \sup_{r, x \in \mathbb{R}_+} \left(\frac{1}{[m_\lambda(I(x,r))]^\beta} \int_{I(x,r)} |f(y)|^p dm_\lambda(y) \right)^{1/p}.$$

4. Outline of Proofs

4.1. Preliminary

Lemma 1(reverse doubling property of m_λ) For any $I \subset \mathbb{R}_+$,

$$\min\left(2, 2^{2\lambda}\right) m_\lambda(I) \le m_\lambda(2I) \le 2^{2\lambda+1} m_\lambda(I). \qquad (1)$$

Lemma 2(Point-wise estimates of $R_{\Delta_\lambda}(y,z)$)

i) There exists a positive constant C such that for any $y, z \in \mathbb{R}_+$ with $y \neq z$,

$$|R_{\Delta_\lambda}(y,z)| \le C \frac{1}{m_\lambda(I(y, |y-z|))}. \qquad (2)$$

ii) There exists a positive constant $\widetilde{C}$ such that for any $y, y_0, z \in \mathbb{R}_+$ with $|y_0 - z| < |y_0 - y|/2$,

$$|R_{\Delta_\lambda}(y, y_0) - R_{\Delta_\lambda}(y, z)| + |R_{\Delta_\lambda}(y_0, y) - R_{\Delta_\lambda}(z, y)| \leq \widetilde{C} \frac{|y_0 - z|}{|y_0 - y|} \frac{1}{m_\lambda(I(y, |y_0 - y|))}. \tag{3}$$

iii) There exist $K_1 \in (0, 1)$ small enough and a positive constant $C_{K_1,\lambda}$ such that for any $y, z \in \mathbb{R}_+$ with $z < K_1 y$,

$$R_{\Delta_\lambda}(y, z) \leq -C_{K_1,\lambda} \frac{1}{y^{2\lambda+1}}.$$

iv) There exist $K_2 \in (1/2, 1)$ such that $1 - K_2$ small enough and a positive constant $C_{K_2,\lambda}$ such that for any $y, z \in \mathbb{R}_+$ with $z/y \in (K_2, 1)$,

$$\left| R_{\Delta_\lambda}(y,z) + \frac{1}{\pi}\frac{1}{y^\lambda z^\lambda}\frac{1}{y-z} \right| \le C_{K_2,\lambda}\frac{1}{y^{2\lambda+1}}\left(\log_+ \frac{\sqrt{yz}}{|y-z|} + 1\right).$$

v) There exists a positive constant C_0 such that for any $y, z \in \mathbb{R}_+$ with $z < y$,

$$R_{\Delta_\lambda}(y,z) \le -C_0 \frac{1}{m_\lambda(I(y, y-z))}.$$

Lemma 3(the characterization of CMO in Bessel setting) Let $f \in \mathrm{BMO}(\mathbb{R}_+, dm_\lambda)$. Then $f \in \mathrm{CMO}(\mathbb{R}_+, dm_\lambda)$ if and only if f satisfies the following three conditions:

(i)

$$\lim_{a \to 0^+} \sup_{m_\lambda(I)=a} M_\lambda(f, I) = 0,$$

(ii)

$$\lim_{a \to \infty} \sup_{m_\lambda(I)=a} M_\lambda(f, I) = 0,$$

(iii)

$$\lim_{R \to \infty} \sup_{I \subset [R, +\infty)} M_\lambda(f, I) = 0.$$

Lemma 4(Fréchet-Kolmogorov's theorem in Bessel setting) Let $1 < p < \infty$, a subset $\mathscr{F}$ of $L^p(\mathbb{R}_+, dm_\lambda)$ is totally bounded (or relatively compact) if and only if the following statements hold:

(a) $\mathscr{F}$ is uniformly bounded, i.e.,

$$\sup_{f\in\mathscr{F}} \|f\|_{L^p(\mathbb{R}_+, dm_\lambda)} < \infty;$$

(b) $\mathscr{F}$ uniformly vanishes at infinity, i.e., for every $\epsilon > 0$, there exists some positive constant M such that for every $f \in \mathscr{F}$,

$$\int_M^\infty |f(x)|^p x^{2\lambda}\, dx < \epsilon^p;$$

(c) $\mathscr{F}$ is uniformly equicontinuous, i.e., for every $\epsilon > 0$, there exists some positive constant ρ, such that for every $f \in \mathscr{F}$ and $y \in \mathbb{R}_+$ with $y < \rho$,

$$\int_0^\infty |f(x+y) - f(x)|^p x^{2\lambda}\, dx < \epsilon^p.$$

Lemma 5(Boundedness of $[b, R_{\Delta_\lambda}]$) Let $b \in \cup_{q>1} L^q_{\mathrm{loc}}(\mathbb{R}_+, dm_\lambda)$ and $p \in (1, \infty)$. Then $b \in \mathrm{BMO}(\mathbb{R}_+, dm_\lambda)$ if and only if $[b, R_{\Delta_\lambda}]$ is bounded on $L^p(\mathbb{R}_+, dm_\lambda)$. Moreover, there exists a positive constant $C \in (1, \infty)$ such that

$$\begin{aligned} C^{-1}\|b\|_{\mathrm{BMO}(\mathbb{R}_+, dm_\lambda)} &\le \left\|[b, R_{\Delta_\lambda}]\right\|_{L^p(\mathbb{R}_+, dm_\lambda)\to L^p(\mathbb{R}_+, dm_\lambda)} \\ &\le C\|b\|_{\mathrm{BMO}(\mathbb{R}_+, dm_\lambda)}. \end{aligned}$$

4.2. Sketch of Proof of Theorem 1

(Sufficiency) **$[b, R_{\Delta_\lambda}]$ is a compact operator on $L^p(\mathbb{R}_+, dm_\lambda) \Rightarrow b \in \mathrm{CMO}(\mathbb{R}_+, dm_\lambda)$:**

- Employing the idea of Uchiyama in [Tôhoku J. Math. 1978].
- Since $[b, R_{\Delta_\lambda}]$ is compact on $L^p(\mathbb{R}_+, dm_\lambda)$, then $[b, R_{\Delta_\lambda}]$ is bounded on $L^p(\mathbb{R}_+, dm_\lambda)$. By Lemma 5, we see that $b \in \mathrm{BMO}(\mathbb{R}_+, dm_\lambda)$.
- To show $b \in \mathrm{CMO}(\mathbb{R}_+, dm_\lambda)$, we use a contradiction argument via Lemma 3. Observe that if $b \notin \mathrm{CMO}(\mathbb{R}_+, dm_\lambda)$, then b does not satisfy at least one of (i)-(iii) in Lemma 3.

$\ast$ Case i): b does not satisfy (i) in Lemma 3 $\Rightarrow$ $[b, R_{\Delta_\lambda}]$ is not compact on $L^p(\mathbb{R}_+, dm_\lambda)$.

∗ Case ii): b does not satisfy (ii) in Lemma 3 $\Rightarrow$ $[b, R_{\Delta_\lambda}]$ is not compact on $L^p(\mathbb{R}_+, dm_\lambda)$.

∗ Case iii): b does not satisfy (iii) in Lemma 3 $\Rightarrow$ $[b, R_{\Delta_\lambda}]$ is not compact on $L^p(\mathbb{R}_+, dm_\lambda)$.

(Necessity) $b \in \mathrm{CMO}(\mathbb{R}_+, dm_\lambda) \Rightarrow [b, R_{\Delta_\lambda}]$ **is compact on** $L^p(\mathbb{R}_+, dm_\lambda)$**:**

- For any $\epsilon > 0$, there exists $b_\epsilon \in \mathscr{D}$ such that

$$\|b - b_\epsilon\|_{\mathrm{BMO}(\mathbb{R}_+, dm_\lambda)} < \epsilon$$

and

$$\|[b, R_{\Delta_\lambda}] - [b_\epsilon, R_{\Delta_\lambda}]\|_{L^p(\mathbb{R}_+, dm_\lambda) \to L^p(\mathbb{R}_+, dm_\lambda)} \lesssim \|b - b_\epsilon\|_{\mathrm{BMO}(\mathbb{R}_+, dm_\lambda)} \lesssim \epsilon.$$

Thus, it suffices to show that $[b, R_{\Delta_\lambda}]$ is a compact operator for $b \in \mathscr{D}$.

• For $b \in \mathscr{D}$, it suffices to show that $[b, R_{\Delta_\lambda}]\mathscr{F}$ is relatively compact for every bounded subset $\mathscr{F} \in L^p(\mathbb{R}_+, dm_\lambda)$.

• Only need to show that $[b, R_{\Delta_\lambda}]\mathscr{F}$ satisfies the conditions (a)—(c) in Lemma 4.

∗ By Lemma 5 and the fact that $b \in \mathrm{BMO}(\mathbb{R}_+, dm_\lambda)$, $[b, R_{\Delta_\lambda}]$ is bounded on $L^p(\mathbb{R}_+, dm_\lambda)$, which implies $[b, R_{\Delta_\lambda}]\mathscr{F}$ satisfies (a) in Lemma 4.

∗ Since $b \in \mathscr{D}$, by (2) and the Hölder inequality, there exists M such that for any $x > M$,

$$\begin{aligned} \left|[b, R_{\Delta_\lambda}]f(x)\right| &\leq |b(x)||R_{\Delta_\lambda}f(x)| + |R_{\Delta_\lambda}(bf)(x)| \\ &\lesssim \|f\|_{L^p(\mathbb{R}_+, dm_\lambda)} \frac{1}{m_\lambda(I(x,x))}. \end{aligned}$$

Hence (b) in Lemma 4 holds for $[b, R_{\Delta_\lambda}]\mathscr{F}$.

* It remains to prove $[b, R_{\Delta_\lambda}]\mathscr{F}$ also satisfies (c).

Let ϵ be a fixed positive constant in $(0, \frac{1}{2})$ and $z \in \mathbb{R}_+$ small enough. Then for any $x \in \mathbb{R}_+$,

$$
\begin{aligned}
&[b, R_{\Delta_\lambda}]f(x) - [b, R_{\Delta_\lambda}]f(x+z) \\
&= \int_0^\infty R_{\Delta_\lambda}(x,y)[b(x) - b(y)]f(y)y^{2\lambda}\,dy \\
&\quad - \int_0^\infty R_{\Delta_\lambda}(x+z,y)[b(x+z) - b(y)]f(y)y^{2\lambda}\,dy \\
&= \int_{|x-y|>\epsilon^{-1}z} R_{\Delta_\lambda}(x,y)[b(x) - b(x+z)]f(y)y^{2\lambda}\,dy \\
&\quad + \int_{|x-y|>\epsilon^{-1}z} [R_{\Delta_\lambda}(x,y) - R_{\Delta_\lambda}(x+z,y)] \\
&\qquad \times [b(x+z) - b(y)]f(y)y^{2\lambda}\,dy
\end{aligned}
$$

$$+\int_{|x-y|\le\epsilon^{-1}z} R_{\Delta_\lambda}(x,y)[b(x)-b(y)]f(y)y^{2\lambda}\,dy$$

$$-\int_{|x-y|\le\epsilon^{-1}z} R_{\Delta_\lambda}(x+z,y)$$

$$\times[b(x+z)-b(y)]f(y)y^{2\lambda}\,dy$$

$$=:\sum_{j=1}^4 \mathrm{L}_i.$$

Then,

$$|\mathrm{L}_1|\le|b(x)-b(x+z)|\sup_{t>0}\left|\int_{|x-y|>t} R_{\Delta_\lambda}(x,y)f(y)y^{2\lambda}\,dy\right|$$

$$=:|b(x)-b(x+z)|R_{\Delta_\lambda\,*}f(x);$$

$$|\mathrm{L}_2|\lesssim|z|\int_{|x-y|>\epsilon^{-1}z}\frac{|f(y)|}{m_\lambda(I(x,|x-y|))|x-y|}y^{2\lambda}\,dy;$$

$$|L_3| \lesssim \int_{|x-y|\le\epsilon^{-1}z} \frac{|x-y|}{m_\lambda(I(x,|x-y|))} |f(y)| y^{2\lambda}\, dy$$

and

$$|L_4| \lesssim \int_{|x-y|\le\epsilon^{-1}z} \frac{|x+z-y|}{m_\lambda(I(x+z,|x+z-y|))} |f(y)| y^{2\lambda}\, dy.$$

Consequently, as b is uniformly continuous, by letting z small enough depending on ϵ, we have that

$$\int_0^\infty |L_1|^p x^{2\lambda}\, dx \lesssim \int_0^\infty \Big[|b(x)-b(x+z)| R_{\Delta_\lambda *} f(x)\Big]^p x^{2\lambda}$$

$$\lesssim \epsilon^p \|f\|^p_{L^p(\mathbb{R}_+,\, dm_\lambda)};$$

$$\int_0^\infty |L_2|^p x^{2\lambda}\,dx$$

$$\lesssim z^p \int_0^\infty \left\{ \left[\int_{|x-y|>\epsilon^{-1}z} \frac{y^{2\lambda}}{|x-y| m_\lambda(I(x,|x-y|))}\,dy \right]^{p/p'} \right.$$

$$\left. \times \int_{|x-y|>\epsilon^{-1}z} \frac{|f(y)|^p}{|x-y| m_\lambda(I(x,|x-y|))} y^{2\lambda}\,dy \right\} x^{2\lambda}\,dx$$

$$\lesssim z^p (\epsilon z^{-1})^{(p/p')+1} \|f\|^p_{L^p(\mathbb{R}_+,\,dm_\lambda)} \lesssim \epsilon^p \|f\|^p_{L^p(\mathbb{R}_+,\,dm_\lambda)};$$

$$
\begin{aligned}
&\int_0^\infty |\mathrm{L}_3|^p x^{2\lambda}\,dx \\
&\lesssim \int_0^\infty \left\{ \left[\int_{|x-y|\le \epsilon^{-1} z} \frac{|x-y|}{m_\lambda(I(x,|x-y|))} y^{2\lambda}\,dy \right]^{p/p'} \right. \\
&\quad \left. \times \int_{|x-y|\le \epsilon^{-1} z} \frac{|x-y||f(y)|^p}{m_\lambda(I(x,|x-y|))} y^{2\lambda}\,dy \right\} x^{2\lambda}\,dx \\
&\lesssim (\epsilon^{-1} z)^p \|f\|_{L^p(\mathbb{R}_+,\,dm_\lambda)}^p;
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^\infty |\mathrm{L}_4|^p x^{2\lambda}\,dx \\
&\lesssim \int_0^\infty \bigg\{ \bigg[\int_{|x+z-y|\le\epsilon^{-1}z+z} \frac{|x+z-y|}{m_\lambda(I(x+z,|x+z-y|))} \\
&\qquad \times y^{2\lambda}\,dy \bigg]^{p/p'} \int_{|x+z-y|\le\epsilon^{-1}z+z} \\
&\qquad\qquad \times \frac{|x+z-y||f(y)|^p}{m_\lambda(I(x+z,|x+z-y|))} y^{2\lambda}\,dy \bigg\} x^{2\lambda}\,dx \\
&\lesssim (\epsilon^{-1}z+z)^p \|f\|^p_{L^p(\mathbb{R}_+,\,dm_\lambda)} \\
&\lesssim (\epsilon^{-1}z)^p \|f\|^p_{L^p(\mathbb{R}_+,\,dm_\lambda)}.
\end{aligned}
$$

Hence,

$$\left[\int_0^\infty \left|[b, R_{\Delta_\lambda}]f(x) - [b, R_{\Delta_\lambda}]f(x+z)\right|^p x^{2\lambda}\,dx\right]^{1/p}$$

$$\lesssim \sum_{i=1}^{4}\left(\int_0^\infty |\mathrm{L}_i|^p x^{2\lambda}\,dx\right)^{1/p} \lesssim \epsilon\|f\|_{L^p(\mathbb{R}_+,\,dm_\lambda)}.$$

This shows that $[b, R_{\Delta_\lambda}]\mathscr{F}$ satisfies the condition (c) in Lemma 4, and completes the proof of Theorem 1.

□

Thank you!