Noncommutative dyadic martingales and Walsh-Fourier series

Yong Jiao

Central South University

jiaoyong@csu.edu.cn

Wuhan University, Wuhan, May 19, 2017

Noncommutative dyadic martingales and Walsh-Fourier series

Yong Jiao

Central South University

jiaoyong@csu.edu.cn

Wuhan University, Wuhan, May 19, 2017

Joint with Dejian Zhou, Lian Wu and Dmitriy Zanin

Overview

1. The dual space of $h_p^r(\mathcal{R})$ for $0 < p < 1$
2. Partial Sums of Noncommutative Walsh-Fourier series
3. Noncommutative Walsh systems in $L_p(\mathcal{R})$ and in $BMO(\mathcal{R})$

Some Classical Backgrounds

Some Classical Backgrounds

- Let $f = (f_n)$ be a martingale defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

The dual space of $h_p^r(\mathcal{R})$ for $0 < p < 1$

Some Classical Backgrounds

- Let $f = (f_n)$ be a martingale defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
- Let H_p^s denote the martingale Hardy space associated with the conditional quadratic variation, that is,

$$H_p^s = \Big\{ f = (f_n)_{n\geq 0} : \|f\|_{H_p^s} = \Big\| \Big(\sum_{i=1}^{\infty} \mathbb{E}_{i-1}|d_i f|^2 \Big)^{\frac{1}{2}} \Big\|_p < \infty \Big\}.$$

- Define the BMO space: $1 \leq r < \infty$

$$BMO_r(\alpha) = \{ f \in L_r : \|f\|_{BMO_r(\alpha)} = \sup_{\nu \in \mathcal{T}} \mathbb{P}(\nu < \infty)^{-\frac{1}{r}-\alpha} \|f - f^{\nu}\|_r < \infty \}$$

Some Classical Backgrounds

Some Classical Backgrounds

- **Weisz (1994, Lecture Math. Note)**

$$\left(H_p^s\right)^* = BMO_2(\alpha), \quad (0 < p < 1,\ \alpha = \frac{1}{p} - 1)$$

Some Classical Backgrounds

- **Weisz (1994, Lecture Math. Note)**

$$\left(H_p^s\right)^* = BMO_2(\alpha), \quad (0 < p < 1,\ \alpha = \frac{1}{p} - 1)$$

- **Nakai et al. (2012, Math. Nachr.)**

$$\left(H_\Phi^s\right)^* = \mathcal{L}_{2,\phi} \quad \phi(r) = \frac{1}{r\Phi^{-1}(1/r)}.$$

Some Classical Backgrounds

- **Weisz (1994, Lecture Math. Note)**

$$\left(H_p^s\right)^* = BMO_2(\alpha), \quad (0 < p < 1,\ \alpha = \frac{1}{p} - 1)$$

- **Nakai et al. (2012, Math. Nachr.)**

$$\left(H_\Phi^s\right)^* = \mathcal{L}_{2,\phi} \quad \phi(r) = \frac{1}{r\Phi^{-1}(1/r)}.$$

- **Jiao et al. (2017, Trans. AMS)**

$$\left(H_{p,q}^s\right)^* = BMO_2(\alpha), \quad (0 < q \leq 1),$$

and

$$\left(H_{p,q}^s\right)^* = BMO_{2,q}(\alpha), \quad (1 < q < \infty),$$

where $0 < p \leq 1$ and $\alpha = \frac{1}{p} - 1$.

The NC Case

The NC Case

- A description of the dual space of $h_p^c(\mathcal{M})$ for $0 < p < 1$ was provided by Bekjan and Chen.

The NC Case

- A description of the dual space of $h_p^c(\mathcal{M})$ for $0 < p < 1$ was provided by Bekjan and Chen.
- However, they pointed out that such description is unsatisfactory. They explicitly asked the following problem:

The dual space of $h_p^r(\mathcal{R})$ for $0 < p < 1$

The NC Case

- A description of the dual space of $h_p^c(\mathcal{M})$ for $0 < p < 1$ was provided by Bekjan and Chen.
- However, they pointed out that such description is unsatisfactory. They explicitly asked the following problem:
  Problem (Bekjan and Chen, 2010, JFA): Can we describe the dual space of $h_p^c(\mathcal{M})$ as a Lipschitz space for $0 < p < 1$?

The NC Case

- A description of the dual space of $h_p^c(\mathcal{M})$ for $0 < p < 1$ was provided by Bekjan and Chen.
- However, they pointed out that such description is unsatisfactory. They explicitly asked the following problem:
  Problem (Bekjan and Chen, 2010, JFA): Can we describe the dual space of $h_p^c(\mathcal{M})$ as a Lipschitz space for $0 < p < 1$?
- In the classical case, the main tool to solve this problem is the constructed atomic decomposition (Wesiz h_p, Jiao etc. $h_{p,q}$, Nakai etc. h_Φ). But so far, this method is unavailable in noncommutative setting.

- Note that Herz considered this problem in classical setting for dyadic martingales **without using atomic decomposition**.

- Note that Herz considered this problem in classical setting for dyadic martingales **without using atomic decomposition**.
  - Let $\mathcal{A}_n$ be the σ-algebra generated by dyadic intervals of length 2^{-n} in the unit interval $[0,1]$, $\mathcal{A}$ be the σ-algebra generated by $\cup_{n\geq 1}\mathcal{A}_n$ and $\mathbb{P}$ denote the Lebesgue measure on $[0,1]$.
  - A martingale $\{f_n\}_{n\geq 1}$ with respect to $([0,1],\mathcal{A},\mathbb{P})$ is called dyadic.
  - Martingale Hardy space and Lipschitz space are respectively defined as

$$\mathsf{h}_p = \Big\{f=(f_n)_{n\geq 0} : \|f\|_{\mathsf{h}_p} = \Big\|\Big(\sum_{n=0}^{\infty}\mathbb{E}_{n-1}(|f_n-f_{n-1}|^2)\Big)^{1/2}\Big\|_p < \infty\Big\};$$

$$\mathrm{Lip}_\alpha = \Big\{f=(f_n)_{n\geq 0}\in L_2 : \|f\|_{\mathrm{Lip}_\alpha} < \infty\Big\},$$

$$\text{where } \|f\|_{\mathrm{Lip}_\alpha} = |\mathbb{E}_0(f)| + \sup_{n\geq 0} 2^{\alpha n}\|\mathbb{E}_n(|f-\mathcal{E}_n f|^2)\|_\infty^{1/2}.$$

  - Herz's result (Herz 1973) can be summarized as follows: let $0<p<1$ and let $\alpha=\frac{1}{p}-1$. Then $(\mathsf{h}_p)^* = \mathrm{Lip}_\alpha$.

- Note that Herz considered this problem in classical setting for dyadic martingales **without using atomic decomposition**.
  - Let $\mathcal{A}_n$ be the σ-algebra generated by dyadic intervals of length 2^{-n} in the unit interval $[0,1]$, $\mathcal{A}$ be the σ-algebra generated by $\cup_{n\geq 1}\mathcal{A}_n$ and $\mathbb{P}$ denote the Lebesgue measure on $[0,1]$.
  - A martingale $\{f_n\}_{n\geq 1}$ with respect to $([0,1],\mathcal{A},\mathbb{P})$ is called dyadic.
  - Martingale Hardy space and Lipschitz space are respectively defined as

$$\mathsf{h}_p = \Big\{ f=(f_n)_{n\geq 0} : \|f\|_{\mathsf{h}_p} = \Big\| \Big(\sum_{n=0}^{\infty} \mathbb{E}_{n-1}(|f_n - f_{n-1}|^2) \Big)^{1/2} \Big\|_p < \infty \Big\};$$

$$\mathrm{Lip}_\alpha = \Big\{ f=(f_n)_{n\geq 0} \in L_2 : \|f\|_{\mathrm{Lip}_\alpha} < \infty \Big\},$$

$$\text{where } \|f\|_{\mathrm{Lip}_\alpha} = |\mathbb{E}_0(f)| + \sup_{n\geq 0} 2^{\alpha n} \|\mathbb{E}_n(|f - \mathcal{E}_n f|^2)\|_\infty^{1/2}.$$

  - Herz's result (Herz 1973) can be summarized as follows: let $0<p<1$ and let $\alpha = \frac{1}{p} - 1$. Then $(\mathsf{h}_p)^* = \mathrm{Lip}_\alpha$.

The dual space of $h_p^r(\mathcal{R})$ for $0 < p < 1$

Question: Although we could not get the duality for generic filtration because of the lack of atomic decompositions, can we obtain the duality for noncommutative dyadic martingales?

The dual space of $\mathrm{h}_p^r(\mathcal{R})$ for $0 < p < 1$

Question: Although we could not get the duality for generic filtration because of the lack of atomic decompositions, can we obtain the duality for noncommutative dyadic martingales?

Noncommutative Dyadic Martingales

The dual space of $h_p^r(\mathcal{R})$ for $0 < p < 1$

Question: Although we could not get the duality for generic filtration because of the lack of atomic decompositions, can we obtain the duality for noncommutative dyadic martingales?

Noncommutative Dyadic Martingales

- $\mathcal{R}$ is the hyperfinite type II_1 factor equipped with its natural increasing filtration $(\mathcal{R}_n)_{n\geq 1}$.

The dual space of $h_p^r(\mathcal{R})$ for $0 < p < 1$

Question: Although we could not get the duality for generic filtration because of the lack of atomic decompositions, can we obtain the duality for noncommutative dyadic martingales?

Noncommutative Dyadic Martingales

- $\mathcal{R}$ is the hyperfinite type II_1 factor equipped with its natural increasing filtration $(\mathcal{R}_n)_{n\geq 1}$.
- $\mathcal{E}_n$ is the conditional expectation of $\mathcal{R}$ onto $\mathcal{R}_n$.

The dual space of $\mathrm{h}_p^r(\mathcal{R})$ for $0 < p < 1$

Question: Although we could not get the duality for generic filtration because of the lack of atomic decompositions, can we obtain the duality for noncommutative dyadic martingales?

Noncommutative Dyadic Martingales

- $\mathcal{R}$ is the hyperfinite type II_1 factor equipped with its natural increasing filtration $(\mathcal{R}_n)_{n\geq 1}$.
- $\mathcal{E}_n$ is the conditional expectation of $\mathcal{R}$ onto $\mathcal{R}_n$.
- A sequence $x = (x_n)_{n\geq 0}$ in $L_1(\mathcal{R})$ is called a sequence of dyadic martingale differences if $\mathcal{E}_{n-1}(x_n) = 0$ for all $n \geq 1$.

The dual space of $h_p^r(\mathcal{R})$ for $0 < p < 1$

Question: Although we could not get the duality for generic filtration because of the lack of atomic decompositions, can we obtain the duality for noncommutative dyadic martingales?

Noncommutative Dyadic Martingales

- $\mathcal{R}$ is the hyperfinite type II_1 factor equipped with its natural increasing filtration $(\mathcal{R}_n)_{n\geq 1}$.
- $\mathcal{E}_n$ is the conditional expectation of $\mathcal{R}$ onto $\mathcal{R}_n$.
- A sequence $x = (x_n)_{n\geq 0}$ in $L_1(\mathcal{R})$ is called a sequence of dyadic martingale differences if $\mathcal{E}_{n-1}(x_n) = 0$ for all $n \geq 1$.
- For $x = (x_n)_{n\geq 0}$ in $L_2(\mathcal{R})$, we define

$$s_{c,n}(x) = \Big(\sum_{k=0}^{n} \mathcal{E}_{k-1}(|x_k|^2)\Big)^{1/2}, \quad s_c(x) = \Big(\sum_{k=0}^{\infty} \mathcal{E}_{k-1}(|x_k|^2)\Big)^{1/2};$$

$$s_{r,n}(x) = \Big(\sum_{k=0}^{n} \mathcal{E}_{k-1}(|x_k^*|^2)\Big)^{1/2}, \quad s_r(x) = \Big(\sum_{k=0}^{\infty} \mathcal{E}_{k-1}(|x_k^*|^2)\Big)^{1/2};$$

- For $0 < p < \infty$, the Hardy space $\mathrm{h}_p^c(\mathcal{R})$ is defined as the collection of all martingale differences $x = (x_n)_{n\geq 0}$ in $L_2(\mathcal{R})$ s.t. $s_c(x) \in L_p(\mathcal{R})$, equipped with the (quasi-)norm $\|(x_n)_{n\geq 0}\|_{\mathrm{h}_p^c} = \|s_c(x)\|_p$.

$$s_{r,n}(x) = \Big(\sum_{k=0}^{n} \mathcal{E}_{k-1}(|x_k^*|^2)\Big)^{1/2}, \quad s_r(x) = \Big(\sum_{k=0}^{\infty} \mathcal{E}_{k-1}(|x_k^*|^2)\Big)^{1/2};$$

- For $0 < p < \infty$, the Hardy space $\mathrm{h}_p^c(\mathcal{R})$ is defined as the collection of all martingale differences $x = (x_n)_{n\geq 0}$ in $L_2(\mathcal{R})$ s.t. $s_c(x) \in L_p(\mathcal{R})$, equipped with the (quasi-)norm $\|(x_n)_{n\geq 0}\|_{\mathrm{h}_p^c} = \|s_c(x)\|_p$.
- For $\alpha \geq 0$, we define the martingale Lipschitz space $\mathrm{Lip}_\alpha^c(\mathcal{R})$ as the set

$$\mathrm{Lip}_\alpha^c(\mathcal{R}) = \{x \in L_2(\mathcal{R}) : \sup_{n\geq 0} 2^{n\alpha}\|\mathcal{E}_n(|x - \mathcal{E}_n x|^2)\|_\infty < \infty\}$$

equipped with the norm

$$\|x\|_{\mathrm{Lip}_\alpha^c(\mathcal{R})} = \max\left(|\tau(x)|, \quad \sup_{n\geq 0} 2^{n\alpha}\|\mathcal{E}_n(|x - \mathcal{E}_n x|^2)\|_\infty^{1/2}\right).$$

Similarly, define $\mathrm{Lip}_\alpha^r(\mathcal{R})$ equipped with the norm

$$\|x\|_{\mathrm{Lip}_\alpha^r(\mathcal{R})} = \|x^*\|_{\mathrm{Lip}_\alpha^c(\mathcal{R})}.$$

Main Result

The following is our main result of this section:

Main Result

The following is our main result of this section:

Theorem 1 (J., Zhou, Wu, Zanin, 2017)

For every $0 < p < 1$, we have $(\mathrm{h}_p^r(\mathcal{R}))^* = \mathrm{Lip}_\alpha^c(\mathcal{R})$, $\alpha = \frac{1}{p} - 1$. More precisely, for every $x \in \mathrm{Lip}_\alpha^c(\mathcal{R})$, the functional

$$\varphi_x : y \mapsto \sum_{n\geq 0} \tau(y_n x_n), \quad y \in \mathrm{h}_p^r(\mathcal{R}),$$

is well-defined and bounded. Conversely, each $\varphi \in (\mathrm{h}_p^r(\mathcal{R}))^*$ is given by the above formula with some $x \in \mathrm{Lip}_\alpha^c(\mathcal{R})$. Moreover,

$$\|x\|_{\mathrm{Lip}_\alpha^c(\mathcal{R})} \leq \|\varphi\|_{(\mathrm{h}_p^r)^*}.$$

Lemma 1 (Junge-Xu, 2003)

Let $0 \leq b \leq a \in L_1(\mathcal{M})$ such that the $\mathrm{supp}(a) = 1$. For $1 \leq k < \infty$, we have

$$\tau\left(a^{\frac{1-k}{2}}(a^k - b^k)a^{\frac{1-k}{2}}\right) \leq 2^k \tau(a-b).$$

Lemma 2 (J., Zhou, Wu, Zanin, 2017)

If $0 \leq b \leq a \in L_1(\mathcal{M})$ and $\beta \in (0,1)$, then

$$\beta\tau(a-b) \leq \tau((a^\beta - b^\beta)a^{1-\beta}) \leq \tau(a-b).$$

Proposition 3 (J., Zhou, Wu, Zanin, 2017)

Let $0 < p < 1$ and let $\alpha = \frac{1}{p} - 1$. If $x \in \mathrm{Lip}_\alpha^c(\mathcal{R})$ and if $y \in \mathrm{h}_p^r(\mathcal{R})$, then

$$\sum_{n\geq 0} \tau(|x_n|^2 s_{r,n}(y)^{2-p}) \leq \frac{2+3p}{p} \|y\|_{\mathrm{h}_p^r}^{2-p} \|x\|_{\mathrm{Lip}_\alpha^c}^2.$$

Proposition 4 (J., Zhou, Wu, Zanin, 2017)

Let $y \in \mathrm{h}_p^r(\mathcal{R})$ for $0 < p \leq 2$. We have

$$\sum_{n\geq 0} \tau\left(|y_n^*|^2 s_{r,n}(y)^{p-2}\right) \leq 2^{\frac{2}{p}} \|y\|_{\mathrm{h}_p^r}^p.$$

Proposition 3 (J., Zhou, Wu, Zanin, 2017)

Let $0 < p < 1$ and let $\alpha = \frac{1}{p} - 1$. If $x \in \mathrm{Lip}_\alpha^c(\mathcal{R})$ and if $y \in \mathrm{h}_p^r(\mathcal{R})$, then

$$\sum_{n\geq 0} \tau(|x_n|^2 s_{r,n}(y)^{2-p}) \leq \frac{2+3p}{p} \|y\|_{\mathrm{h}_p^r}^{2-p} \|x\|_{\mathrm{Lip}_\alpha^c}^2.$$

Proposition 4 (J., Zhou, Wu, Zanin, 2017)

Let $y \in \mathrm{h}_p^r(\mathcal{R})$ for $0 < p \leq 2$. We have

$$\sum_{n\geq 0} \tau\left(|y_n^*|^2 s_{r,n}(y)^{p-2}\right) \leq 2^{\frac{2}{p}} \|y\|_{\mathrm{h}_p^r}^p.$$

Observation: For $x \in \mathcal{R}_n^+$, there exists a projection e such that $\tau(e) = 2^{-n}$ and

$$\|x\|_\infty = 2^n \tau(ex).$$

Outline of the proof: (i) Let $x \in \mathrm{Lip}_\alpha^c(\mathcal{R})$ and $y \in \mathsf{h}_p^r(\mathcal{R})$. Assume that $s_{r,n}(y)$ is invertible. Then

$$\begin{aligned}\varphi_x(y) &\triangleq= \Big|\sum_{n\geq 0}\tau(x_n y_n)\Big| \leq \sum_{n\geq 0}\left\|x_n s_{r,n}(y)^{1-\frac{p}{2}}\right\|_2 \cdot \left\|s_{r,n}(y)^{\frac{p}{2}-1}y_n\right\|_2 \\ &\leq \Big(\sum_{n\geq 0}\left\|x_n s_{r,n}(y)^{1-\frac{p}{2}}\right\|_2^2\Big)^{\frac{1}{2}} \cdot \Big(\sum_{n\geq 0}\left\|s_{r,n}(y)^{\frac{p}{2}-1}y_n\right\|_2^2\Big)^{\frac{1}{2}} \\ &\triangleq \mathrm{I}\cdot \mathrm{II}.\end{aligned}$$

It follows from Proposition 3 that

$$\mathrm{I}^2 = \tau\Big(\sum_{n\geq 0}|x_n|^2 s_{r,n}(y)^{2-p}\Big) \leq \frac{2+3p}{p}\|y\|_{\mathsf{h}_p^r}^{2-p}\|x\|_{\mathrm{Lip}_\alpha^c}^2.$$

It follows from Proposition 4 that

$$\mathrm{II}^2 = \tau\Big(\sum_{n\geq 0}|y_n^*|^2 s_{r,n}(y)^{p-2}\Big) \leq 2^{\frac{2}{p}}\|y\|_{\mathsf{h}_p^r}^p.$$

(ii) Let $\varphi \in (\mathrm{h}_p^r(\mathcal{R}))^*$ such that $\|\varphi\|_{(\mathrm{h}_p^r)^*} \leq 1$.

- Since $L_2(\mathcal{R})$ is dense in $\mathrm{h}_p^r(\mathcal{R})$, there exists $x \in L_2(\mathcal{R})$ such that

$$\varphi(y) = \sum_k \tau(x_k y_k), \quad \forall y = (y_n)_{n\geq 0} \in L_2(\mathcal{R}).$$

It suffices to show that $\|x\|_{\mathrm{Lip}_\alpha^c} \leq \|\varphi\|_{(\mathrm{h}_p^r)^*}$.

- Fix n, choose $e \in \mathcal{R}_n$ such that $\tau(e) = 2^{-n}$ and

$$\|\mathcal{E}_n(|x - \mathcal{E}_n(x)|^2)\|_\infty = 2^n \tau(e \cdot \mathcal{E}_n(|x - \mathcal{E}_n(x)|^2)) = 2^n \sum_{k>n} \tau(e|x_k|^2).$$

- Set

$$y_k = \begin{cases} 0, & k \leq n \\ 2^{n(1+2\alpha)} e x_k^*, & k > n \end{cases}.$$

- Then we have

$$\mathrm{III_n} \triangleq 2^{2n\alpha}\|\mathcal{E}_n(|x-\mathcal{E}_n(x)|^2)\|_\infty = |\sum_k \tau(y_k x_k)|$$

$$= |\varphi(y)| \le \|y\|_{\mathsf{h}_p^r} = \|s_r^2(y)\|_{\frac{p}{2}}^{\frac{1}{2}};$$

$$\mathrm{V_n} \triangleq \mathcal{E}_n(s_r^2(y)) = 2^{2n(1+2\alpha)} \cdot \sum_{k>n} \mathcal{E}_n(e|x_k|^2 e)$$

$$= 2^{2n(1+\alpha)} \cdot e \cdot 2^{2n\alpha}\mathcal{E}_n(|x-\mathcal{E}_n(x)|^2) \cdot e.$$

- Observe that $\mathrm{V_n} \le \mathrm{III_n} \cdot 2^{\mathrm{2n(1+\alpha)}}\mathrm{e}$. This implies

$$\mathrm{III_n} \le \|\mathrm{s_r^2(y)}\|_{\frac{\mathrm{p}}{2}}^{\frac{1}{2}} \le \|\mathrm{V_n^{\frac{1}{2}}}\|_\mathrm{p} \le \mathrm{III_n^{\frac{1}{2}}} \cdot 2^{\frac{\mathrm{n}}{\mathrm{p}}}\|\mathrm{e}\|_\mathrm{p} = \mathrm{III_n^{\frac{1}{2}}}.$$

- Taking the supremum over all $n \ge 0$, we conclude that

$$\sup_{n\ge0} 2^{n\alpha}\|\mathcal{E}_n(|x-\mathcal{E}_n(x)|^2)\|_\infty^{1/2} = \sup_{n\ge0} \mathrm{III_n^{1/2}} \le 1. \qquad \square$$

Classical result due to Watari

- Let $(r_k)_{k\geq 0}$ be the Rademacher functions on $[0, 1)$.

Classical result due to Watari

- Let $(r_k)_{k\geq 0}$ be the Rademacher functions on $[0,1)$.
- The Walsh system $\{\omega_n\}_{n\geq 1}$ is defined as follows: let $\omega_0(t)=1$ and $\omega_n(t)=\prod_{i=1}^{j} r_{n(i)}(t)$, where $n=\sum_{i=1}^{j} 2^{n(i)-1}$ with $\{n(1)>n(2)>\cdots>n(j)>0\}$.

Classical result due to Watari

- Let $(r_k)_{k\geq 0}$ be the Rademacher functions on $[0,1)$.
- The Walsh system $\{\omega_n\}_{n\geq 1}$ is defined as follows: let $\omega_0(t)=1$ and $\omega_n(t)=\prod_{i=1}^{j} r_{n(i)}(t)$, where $n=\sum_{i=1}^{j} 2^{n(i)-1}$ with $\{n(1)>n(2)>\cdots>n(j)>0\}$.
- Every $f\in L_1([0,1))$ can be written as a formal Walsh-Fourier series,

$$f\sim\sum_{k=0}^{\infty}\hat{f}(k)\omega_k, \qquad \hat{f}(k)=\int_0^1 f(t)\omega_k(t)\,dt.$$

Partial Sums of Noncommutative Walsh-Fourier series

Classical result due to Watari

- Let $(r_k)_{k\geq 0}$ be the Rademacher functions on $[0,1)$.
- The Walsh system $\{\omega_n\}_{n\geq 1}$ is defined as follows: let $\omega_0(t)=1$ and $\omega_n(t)=\prod_{i=1}^{j} r_{n(i)}(t)$, where $n=\sum_{i=1}^{j} 2^{n(i)-1}$ with $\{n(1)>n(2)>\cdots>n(j)>0\}$.
- Every $f\in L_1([0,1))$ can be written as a formal Walsh-Fourier series,

$$f\sim\sum_{k=0}^{\infty}\hat{f}(k)\omega_k,\qquad \hat{f}(k)=\int_0^1 f(t)\omega_k(t)\,dt.$$

- The n-th partial sum of the Walsh-Fourier series of f is defined by setting

$$S_n(f)=\sum_{k=0}^{n-1}\hat{f}(k)\omega_k.$$

Then Watari's result reads as follows:

Theorem (Watari, 1964)

For every $f \in L_1(0,1)$ and for every $n \geq 0$, we have

$$\|S_n(f)\|_{L_{1,\infty}(0,1)} \leq c\|f\|_{L_1(0,1)}.$$

Then Watari's result reads as follows:

Theorem (Watari, 1964)

For every $f \in L_1(0,1)$ and for every $n \geq 0$, we have

$$\|S_n(f)\|_{L_{1,\infty}(0,1)} \leq c\|f\|_{L_1(0,1)}.$$

Question:

Can we find a noncommutative analogous of the above result?

Partial Sums of Noncommutative Walsh-Fourier series

γ-th partial sum of noncommutative Walsh-Fourier series

1) Consider the Pauli matrices:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$$

The "anticommutative Rademacher system" in $\mathcal{R}$ is defined by setting:

$$r_0 = 1, \quad r_1 = \sigma_1 \otimes \bigotimes_{i=2}^{\infty} 1_{\mathbb{M}_2}, \quad r_2 = \sigma_2 \otimes \bigotimes_{i=2}^{\infty} 1_{\mathbb{M}_2},$$

and for $n = 2k + s$ where $s \in \{1, 2\}$,

$$r_n = \Big(\bigotimes_{i=1}^{k} \sigma_0 \Big) \bigotimes \sigma_s \bigotimes \Big(\bigotimes_{i=k+2}^{\infty} 1_{\mathbb{M}_2} \Big).$$

The r_n's are self-adjoint unitary operators. Moreover,

$$r_j r_k + r_k r_j = 0, \quad k \neq j.$$

The r_n's are self-adjoint unitary operators. Moreover,

$$r_j r_k + r_k r_j = 0, \quad k \neq j.$$

2) Let $\mathcal{F}$ be the family of all finite subsets of $\mathbb{N}$. For $\gamma \in \mathcal{F}$, the noncommutative Walsh system is defined by setting

$$\omega_\gamma = r_{\gamma(1)} r_{\gamma(2)} \dots r_{\gamma(k)} \text{ for } \gamma = \{\gamma(1) > \gamma(2) > \cdots > \gamma(k) > 0\}.$$

We also set $\omega_\emptyset = 1$.

The r_n's are self-adjoint unitary operators. Moreover,

$$r_j r_k + r_k r_j = 0, \quad k \neq j.$$

2) Let $\mathcal{F}$ be the family of all finite subsets of $\mathbb{N}$. For $\gamma \in \mathcal{F}$, the noncommutative Walsh system is defined by setting

$$\omega_\gamma = r_{\gamma(1)} r_{\gamma(2)} \dots r_{\gamma(k)} \text{ for } \gamma = \{\gamma(1) > \gamma(2) > \cdots > \gamma(k) > 0\}.$$

We also set $\omega_\emptyset = 1$.

3) For $\gamma = \{\gamma(1) > \gamma(2) > \cdots > \gamma(s) > 0\} \in \mathcal{F}$, let $k_\gamma = \sum_{i=1}^{s} 2^{\gamma(i)-1}$.

Definition

For $x \in L_1(\mathcal{R})$, we define the γ-th partial sum of the Walsh-Fourier series of x with a given $\gamma \in \mathcal{F}$ as follows:

$$S_\gamma(x) = \sum_{\{\eta \in \mathcal{F} : k_\eta < k_\gamma\}} \hat{x}(\eta) \omega_\eta.$$

Partial Sums of Noncommutative Walsh-Fourier series

A weak type inequality of γ-th partial sum operator

Partial Sums of Noncommutative Walsh-Fourier series

A weak type inequality of γ-th partial sum operator

The following is our main result of this section:

Theorem 2 (J., Zhou, Wu, Zanin, 2017)

For every $x \in L_1(\mathcal{R})$ and for every $\gamma \in \mathcal{F}$, we have

$$\|S_\gamma(x)\|_{L_{1,\infty}(\mathcal{R})} \leq c\|x\|_{L_1(\mathcal{R})}.$$

A weak type inequality of γ-th partial sum operator

The following is our main result of this section:

Theorem 2 (J., Zhou, Wu, Zanin, 2017)

For every $x \in L_1(\mathcal{R})$ and for every $\gamma \in \mathcal{F}$, we have

$$\|S_\gamma(x)\|_{L_{1,\infty}(\mathcal{R})} \leq c\|x\|_{L_1(\mathcal{R})}.$$

Corollary (J., Zhou, Wu, Zanin, 2017)

For every $x \in L_p(\mathcal{R})$, $1 < p < \infty$, and for every $\gamma \in \mathcal{F}$, we have

$$\|S_\gamma(x)\|_{L_p(\mathcal{R})} \leq c_p\|x\|_{L_p(\mathcal{R})}.$$

Corollary (J., Zhou, Wu, Zanin, 2017)

For every $x \in \Lambda_\psi(\mathcal{R})$ and for every$\gamma \in \mathcal{F}$, we have

$$\|S_\gamma(x)\|_{L_1(\mathcal{R})} \leq c_{abs}\|x\|_{\Lambda_\psi(\mathcal{R})}.$$

Here, Λ_ψ is a Lorentz space with $\psi(t) = t(1 + \log(\frac{1}{t}))$.

NC Calderon-Zygmund decomposition (Parcet, 2003, JFA)

1) For $0 \leq x \in L_1(\mathcal{R})$ and $\lambda > 0$, it follows from noncommutative Calderón-Zygmund decomposition that $x = A + B$, where

$$A = qxq + \sum_{k=1}^{\infty} p_k \mathcal{E}_k x p_k;$$

$$B = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} p_k(x - \mathcal{E}_{k\vee l}x)p_l + \sum_{1\leq k<l}^{\infty} p_k(\mathcal{E}_l x - \mathcal{E}_{l-1}x)q_{l-1} + q_{l-1}(\mathcal{E}_l x - \mathcal{E}_{l-1}x)p_k.$$

NC Calderon-Zygmund decomposition (Parcet, 2003, JFA)

1) For $0 \leq x \in L_1(\mathcal{R})$ and $\lambda > 0$, it follows from noncommutative Calderón-Zygmund decomposition that $x = A + B$, where

$$A = qxq + \sum_{k=1}^{\infty} p_k \mathcal{E}_k x p_k;$$

$$B = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} p_k (x - \mathcal{E}_{k \vee l} x) p_l + \sum_{1 \leq k < l}^{\infty} p_k (\mathcal{E}_l x - \mathcal{E}_{l-1} x) q_{l-1} + q_{l-1} (\mathcal{E}_l x - \mathcal{E}_{l-1} x) p_k.$$

Lemma 3

If $0 \leq x \in L_1(\mathcal{R})$ and $\lambda > 0$, then

$$\|A\|_2^2 \leq 16 \lambda \|x\|_1.$$

2) For $0 \leq x \in L_1(\mathcal{R})$ and $\delta \in \mathcal{F}$, define the operator D_δ by setting

$$D_\delta(x) = \sum_{\alpha \in \mathcal{F}, \alpha(1) \in \delta} \hat{x}(\alpha)\omega_\alpha.$$

2) For $0 \leq x \in L_1(\mathcal{R})$ and $\delta \in \mathcal{F}$, define the operator D_δ by setting

$$D_\delta(x) = \sum_{\alpha \in \mathcal{F}, \alpha(1) \in \delta} \hat{x}(\alpha)\omega_\alpha.$$

Lemma 4 (J., Zhou, Wu, Zanin, 2017)

Let $\delta \in \mathcal{F}$ and let $x \in L_1(\mathcal{R})$.

(i) For every $k, l \geq 1$, we have

$$D_\delta(p_k(x - \mathcal{E}_{k \vee l}x)p_l) = p_k D_\delta(x - \mathcal{E}_{k \vee l}x)p_l.$$

(ii) For every $1 \leq k < l$, we have

$$D_\delta(p_k(\mathcal{E}_l x - \mathcal{E}_{l-1}x)q_{l-1}) = p_k D_\delta(\mathcal{E}_l x - \mathcal{E}_{l-1}x)q_{l-1},$$

$$D_\delta(q_{l-1}(\mathcal{E}_l x - \mathcal{E}_{l-1}x)p_k) = q_{l-1} D_\delta(\mathcal{E}_l x - \mathcal{E}_{l-1}x)p_k.$$

3) Sketch of the proof of Theorem 2:

- Let $\delta \in \mathcal{F}$. For $x \in L_1(\mathcal{R})$, we have $S_\delta(x)\omega_\delta^* = D_\delta(x\omega_\delta^*)$. Since $|\omega_\delta S_\delta(x)| = |S_\delta(x)|$, it suffices to prove the weak inequality for D_δ.
- For $\lambda > 0$, using the noncommutative Calderón-Zygmund's decomposition we have

$$\tau\Big(\chi_{(\lambda,\infty)}(|D_\delta(x)|)\Big) \leq \tau\Big(\chi_{(\lambda,\infty)}(|D_\delta(A)|)\Big) + \tau\Big(\chi_{(0,\infty)}(|D_\delta(B)|)\Big) = \mathrm{V} + \mathrm{VI}.$$

- From Lemma 3 we have

$$\mathrm{V} \leq \lambda^{-2}\|\mathrm{D}_\delta(\mathrm{A})\|_2^2 \leq \lambda^{-2}\|\mathrm{A}\|_2^2 \leq \frac{16}{\lambda}\|\mathrm{x}\|_1.$$

- By Lemma 4,

$$\mathrm{VI} \leq 3\tau(1-\mathrm{q}) \leq \frac{3}{\lambda}\|\mathrm{x}\|_1. \qquad \square$$

Let $W_L = \{\gamma \in \mathcal{F} : \ \mathrm{Card}(\gamma) = L\}$. Denote the closed linear space of $(\omega_\gamma)_{\gamma \in W_L}$ by $[W_L]$.

Noncommutative Walsh systems in $L_p(\mathcal{R})$ and in $BMO(\mathcal{R})$

Let $W_L = \{\gamma \in \mathcal{F} : \mathrm{Card}(\gamma) = L\}$. Denote the closed linear space of $(\omega_\gamma)_{\gamma \in W_L}$ by $[W_L]$.

Theorem (Müller and Schechtman, 1989)

For $L \geq 2$, $[W_L]$ is not complemented in martingale Hardy space H_1.

Noncommutative Walsh systems in $L_p(\mathcal{R})$ and in $BMO(\mathcal{R})$

Let $W_L = \{\gamma \in \mathcal{F} : \ \mathrm{Card}(\gamma) = L\}$. Denote the closed linear space of $(\omega_\gamma)_{\gamma \in W_L}$ by $[W_L]$.

Theorem (Müller and Schechtman, 1989)

For $L \geq 2$, $[W_L]$ is not complemented in martingale Hardy space H_1.

Our main result of this section demonstrates a very substantial difference from the above theorem.

Noncommutative Walsh systems in $L_p(\mathcal{R})$ and in $BMO(\mathcal{R})$

Let $W_L = \{\gamma \in \mathcal{F} : \mathrm{Card}(\gamma) = L\}$. Denote the closed linear space of $(\omega_\gamma)_{\gamma \in W_L}$ by $[W_L]$.

Theorem (Müller and Schechtman, 1989)

For $L \geq 2$, $[W_L]$ is not complemented in martingale Hardy space H_1.

Our main result of this section demonstrates a very substantial difference from the above theorem.

Theorem 3 (J., Zhou, Wu, Zanin, 2017)

$[W_2]$ is complemented in $\mathrm{bmo}(\mathcal{R})$ and in $\mathcal{H}_1(\mathcal{R})$.

Theorem 4

Let $1 \leq p < \infty$ and let $L \geq 1$. For every $(\alpha_\gamma)_{\gamma \in \mathcal{F}} \in l_2(\mathcal{F})$, we have

$$\Big\| \sum_{\gamma \in W_L} \alpha_\gamma \omega_\gamma \Big\|_{L_p(\mathcal{R})} \approx_{p,L} \Big(\sum_{\gamma \in W_L} |\alpha_\gamma|^2 \Big)^{1/2}.$$

Theorem 4

Let $1 \leq p < \infty$ and let $L \geq 1$. For every $(\alpha_\gamma)_{\gamma\in\mathcal{F}} \in l_2(\mathcal{F})$, we have

$$\Big\| \sum_{\gamma\in W_L} \alpha_\gamma \omega_\gamma \Big\|_{L_p(\mathcal{R})} \approx_{p,L} \Big(\sum_{\gamma\in W_L} |\alpha_\gamma|^2 \Big)^{1/2}.$$

Theorem 5

For every $(\alpha_\gamma)_{\gamma\in\mathcal{F}} \in l_2(\mathcal{F})$, we have

$$\Big(\sum |\alpha_\gamma|^2 \Big)^{1/2} \leq \Big\| \sum_{\gamma\in W_2} \alpha_\gamma \omega_\gamma \Big\|_{\mathrm{bmo}^c} \leq 2 \Big(\sum |\alpha_\gamma|^2 \Big)^{1/2}.$$

Proof of Theorem 3: Let P be the orthogonal projection on $L_2(\mathcal{R})$ by setting

$$Px = \sum_{\gamma \in W_2} \langle x, \omega_\gamma \rangle \omega_\gamma, \quad x \in L_2(\mathcal{R}).$$

According to Theorem 5, we have

$$\|Px\|_{\mathrm{bmo}^c(\mathcal{R})} \leq 4\|Px\|_2 \leq 4\|x\|_2 \leq 4\|x\|_{\mathrm{bmo}^c(\mathcal{R})}.$$

The same inequality holds in $\mathrm{bmo}^r(\mathcal{R})$ and, therefore, in $\mathrm{bmo}(\mathcal{R})$. Thus, $[W_2]$ is complemented in $\mathrm{bmo}(\mathcal{R})$. Recall that $\mathcal{BMO}^c(\mathcal{R}) = \mathrm{bmo}^c(\mathcal{R})$. By duality,

$$\|Px\|_{\mathcal{H}_1^c(\mathcal{R})} \lesssim \|x\|_{\mathcal{H}_1^c(\mathcal{R})}.$$

Therefore,

$$\begin{aligned}\|Px\|_{\mathcal{H}_1(\mathcal{R})} &\leq \inf\left\{\|Py\|_{\mathcal{H}_1^c(\mathcal{R})} + \|Pz\|_{\mathcal{H}_1^r(\mathcal{R})} : \ x = y + z\right\} \\ &\lesssim \inf\left\{\|y\|_{\mathcal{H}_1^c(\mathcal{R})} + \|z\|_{\mathcal{H}_1^r(\mathcal{R})} : \ x = y + z\right\} = \|x\|_{\mathcal{H}_1(\mathcal{R})}. \qquad \square\end{aligned}$$

Thank You !