Similarity degree of twisted group algebras

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Harmonic analysis on groups

- G **is a locally compact group** with left Haar measure m. Let $L_p(G) = L_p(G, m)$ for $1 \leq p \leq \infty$. $L_1(G)$ is called the **group algebra** and is a Banach $*$-algebra under convolution

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and involution

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• **Representation :** $\pi : G \to \mathcal{U}(\mathcal{H})$ is a continuous homomorphism. It induces an involutive representation of $L_1(G)$:

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- **Left regular representation :** $\lambda : G \to \mathcal{U}(L_2(G))$ given by

$$\lambda(s)g(t) = g(s^{-1}t).$$

C^*-algebras and von Neumann algebras

- $C^*(G)$ is the enveloping C^*-algebra of G with norm

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$$C^*(G) = \overline{L_1(G)}^{\|\cdot\|_*}.$$

In general, denote $C^*_\pi(G) = \overline{\pi(L_1(G))}^{\|\cdot\|}$.

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$$VN(G) = VN_\lambda(G), \quad C^*_r(G) = C^*_\lambda(G).$$

Fourier algebras

- $B(G)$ is the **Fourier-Stieltjes algebra** given by

$$B(G) = \{\pi_{\xi,\eta}(\cdot) = \langle \pi(\cdot)\xi|\eta\rangle : \pi \text{ continuous unitary repr., } \xi, \eta \in \mathcal{H}_\pi\}.$$

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- $B(G)$ is a Banach algebra under pointwise multiplication. $A(G)$ is an ideal of $B(G)$, so is also a Banach algebra.

Multipliers

• **A multiplier** of $A(G)$ is a function $u : G \to \mathbb{C}$ such that $uv \in A(G)$ for all $v \in A(G)$. If $m_u : v \mapsto uv$ is bounded on $A(G)$ we say $u \in \mathbf{MA(G)}$. If m_u is completely bounded we say $u \in \mathbf{M_{cb}A(G)}$.

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• Passing to the dual side.

$$\begin{aligned} M_u : \quad & VN(G) \to VN(G) \\ & \lambda(s) \mapsto u(s)\lambda(s) \end{aligned}$$

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$$\varphi \in M_{cb}A(G) \quad \text{iff} \quad \varphi(t^{-1}s) = \langle \xi(s) | \eta(t) \rangle \text{ for some } \xi, \eta : G \to \mathcal{H}.$$

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• $B(G) \subset M_{cb}A(G) \subset MA(G)$.

Amenability

- G **is called amenable** if there exists a left invariant state m on $L_\infty(G)$.

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- **Equivalent conditions :**
  1. G satisfies the Folner condition ;
  2. $A(G)$ has b.a.i. ;
  3. The trivial homomorphism $\varphi : \lambda(s) \mapsto 1$ is a state on $C_r^*(G)$;
  4. $(h_\alpha) \subset S_{L_2(G)}$ compactly supported, s.t. $\|\lambda(s)h_\alpha - h_\alpha\| \to 0$, $\forall s \in G$;
  5. $C^*(G) = C_r^*(G)$ isometrically ;
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  5. $C^*(G) = C_r^*(G)$ isometrically ;
  6. e.t.c.
- G is amenable $\Rightarrow$ $B(G) = MA(G)$. Conversely :
  1. Discrete $B(G) = MA(G)$ $\Rightarrow$ amenability (By Nebbia)
  2. Discrete $B(G) = M_{cb}A(G)$ $\Rightarrow$ amenability (By Bożejko)
  3. Non discrete $B(G) = MA(G)$ $\Rightarrow$ amenability (By Losert)
  4. Non discrete $B(G) = M_{cb}A(G)$ $\Rightarrow$ amenability (known by Losert, Ruan, but unpublished)

Similarity problem

• **Framework :** Triple $\mathcal{A}, E, \iota$, with $\mathcal{A}$ unital algebra , $E \subset \mathcal{A}$ linear subspace with o.s.s., $\iota : E \to \mathcal{A}$ the inclusion mapping. $\mathcal{A}$ can be equipped with an o.s.s. : For any $c \geq 1$, consider the family $\mathcal{C}_c$ of all unital homomorphisms $u : \mathcal{A} \to B(\mathcal{H}_u)$ such that $\|u\iota\|_{cb} \leq c$. Then we equip $\mathcal{A}$ with the norm $\|a\|_c = \sup_{u \in \mathcal{C}_c} \|u(a)\|$, denote by $\widetilde{\mathcal{A}}_c$ the completion.

• **Similarity problem :**

1. Is every unital homomorphism $u : \mathcal{A} \to B(\mathcal{H})$ such that $\|u\iota\|_{cb} < \infty$ is similar to a homomorphism v satisfying $\|v\iota\|_{cb} \leq 1$?
2. Equivalently, does there exist an invertible $S : \mathcal{H} \to \mathcal{H}$ such that the map $u_S : a \mapsto Su\iota(a)S^{-1}$ from E to $B(\mathcal{H})$ is completely contractive ?
3. If S exists, what are smallest $K > 0$ and $\alpha \geq 0$ such that $\|S\|\|S^{-1}\| \leq K\|u\iota\|_{cb}^{\alpha}$? α : similarity degree.

Theorem

- **For group algebra :** $\mathcal{A} = (L_1(G), *)$ and $E = L_1(G)$ natural o.s.s. and ι is identity map. The result is

Theorem

Let G be a locally compact group. Any bounded homomorphism of the group algebra $(L_1(G), *)$ to the algebra of bounded operators on a Hilbert space $\pi : (L_1(G), *) \to B(\mathcal{H}_\pi)$, admits an invertible S in $B(\mathcal{H}_\pi)$ for which

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- **Proof by Pisier (1998 discrete), Spronk (thesis, non discrete) :**

(a) If G is amenable, S is constructed by using the left invariant mean.

(b) If there exists S satisfying (i) (ii) above, one can prove $B(G) = M_{cb}A(G)$, then the amenability follows.

Twisted algebras

• **Normalized** 2-**cocycle** on G with values in $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is a map $\sigma : G \times G \to \mathbb{T}$ such that

$$\sigma(s,t)\,\sigma(st,r) = \sigma(t,r)\sigma(s,tr) \ \ \forall\, s,t,r \in G,$$

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• σ-**repr.** $\pi : G \to \mathcal{U}(\mathcal{H})$ such that $\pi(s)\pi(t) = \sigma(s,t)\pi(st)$.
• **Left regular** σ**-repr.** $\lambda_\sigma : G \to \mathcal{U}(L_2(G))$ such that

$$\lambda(s)g(t) = \sigma(s, s^{-1}t)\, g(s^{-1}t).$$

algebras

- $C^*(G,\sigma)$ enveloping C^*-alg. $\sup\{\|\pi(f)\| : \pi$ is $\sigma -$ representation$\}$.
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- $VN(G,\sigma) = VN_{\lambda_\sigma}(G)$, $C^*_r(G,\sigma) = C^*_{\lambda_\sigma}(G)$.
- **Fourier spaces**
  1. $B(G,\sigma) = \{\pi_{\xi,\eta}(\cdot) = \langle \pi(\cdot)\xi|\eta\rangle : \pi\ \sigma -$ repr., $\xi,\eta \in \mathcal{H}_\pi\}$.
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Proposition

If $\varphi_1 \in B(G,\sigma_1)$ and $\varphi_2 \in B(G,\sigma_2)$, then $\varphi_1\varphi_2 \in B(G,\sigma_1\sigma_2)$ and

$$\|\varphi_1\varphi_2\|_{B(G,\sigma_1\sigma_2)} \leq \|\varphi_1\|_{B(G,\sigma_1)}\|\varphi_2\|_{B(G,\sigma_2)}.$$

Multipliers

- **A multiplier** from $A(G)$ to $A(G,\sigma)$ is a function $u : G \to \mathbb{C}$ such that $uv \in A(G,\sigma)$ for all $v \in A(G)$. If $m_u : v \mapsto uv$ is bounded we say $u \in M(A(G), A(G,\sigma))$. If m_u is completely bounded we say $u \in M_{cb}(A(G), A(G,\sigma))$.

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- **Characterization of $M_{cb}(A(G), A(G,\sigma))$:**

$$\varphi \in M_{cb}(A(G), A(G,\sigma)) \Leftrightarrow \overline{\sigma(t,t^{-1}s)}\varphi(t^{-1}s) = \langle \xi(s) | \eta(t) \rangle$$

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- $B(G,\sigma) \subset M_{cb}(A(G), A(G,\sigma)) \subset M(A(G), A(G,\sigma))$.

Theorem

Let G be locally compact group.

1. *If the group G is amenable, then $B(G,\sigma) = M\big(A(G), A(G,\sigma)\big)$.*
2. *If $B(G,\sigma) = M_{cb}\big(A(G), A(G,\sigma)\big)$, then G is amenable.*

Similarity of twisted algebras

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Let G be a locally compact group. Any bounded homomorphism of the twisted group algebra $(L_1(G), *_\sigma)$ to the algebra of bounded operators on a Hilbert space $\pi : (L_1(G), *_\sigma) \to B(\mathcal{H}_\pi)$, admits an invertible S in $B(\mathcal{H}_\pi)$ for which

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- **Proof :**

(a) If G is amenable, S is constructed by using the left invariant mean.

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Thank you for listening!