Weighted estimates for the multilinear maximal function on the upper half-spaces

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Classical Hardy-littlewood maximal function on $\mathbb{R}^n$

- Let $\mathbb{R}^n$ be the n-dimensional real Euclidean space and f a real valued measurable function, the classical Hardy-littlewood maximal function is defined by

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)|dy,$$

where Q is a cube with its sides parallel to the coordinate axes and $|Q|$ is the Lebesgue measure of Q.

- Let u, v be two weights. Muckenhoupt (1972) showed that

$$\begin{cases} M : L^p(v,\mathbb{R}^n) \to L^{p,\infty}(u,\mathbb{R}^n) & \text{iff } (u,v) \in A_p, \text{ where } p \geq 1; \\ M : L^p(v,\mathbb{R}^n) \to L^p(v,\mathbb{R}^n) & \text{iff } v \in A_p, \text{ where } p > 1 \end{cases}$$

- Let $p > 1$, Sawyer (1982) gave the testing condition and characterized the weights for which M is bounded from $L^p(v,\mathbb{R}^n)$ to $L^p(u,\mathbb{R}^n)$.
- Motivated by Muckenhoupt (1972) and Sawyer (1982), the theory of weighted inequalities developed rapidly, not only for the Hardy–Littlewood maximal operator but also for some of the main operators in Harmonic Analysis like Calderón–Zygmund operators.

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Multilinear Hardy-littlewood maximal function on $\mathbb{R}^n$

- Recently, a large body of literature on the topic of multilinear weighted norm inequalities appeared. This study is based on multiple simultaneous decompositions and is naturally more complicated than its linear counterpart, but is also more far-reaching and yields more flexible results.
- The new multilinear maximal function

$$\mathcal{M}(f_1, \ldots, f_m)(x) := \sup_{x \in Q} \prod_{i=1}^{m} \frac{1}{|Q|} \int_Q |f_i(y_i)| dy_i, \quad x \in \mathbb{R}^n$$

associated with cubes with sides parallel to the coordinate axes was first defined and the corresponding weight theory was studied in Lerner, Ombrosi, Pérez, Torres and Trujillo-González(2009).

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- The relevant class of multiple weights for $\mathcal{M}$ is given by the condition $A_{\vec{p}}$ in Lerner, Ombrosi, Pérez, Torres and Trujillo-González(2009).
- Using a dyadic discretization technique, Damián, Lerner and Pérez (2015) and Li, Moen and Sun (2014) proved some sharp weighted norm inequalities for the multilinear maximal operator $\mathcal{M}$.

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Multilinear weight $S_{\vec{p}}$ and weighted inequalities

- In order to establish the generalization of Sawyer's theorem to the multilinear setting, Chen and Damián (2013) used a reverse Hölder's condition $RH_{\vec{p}}$ on the weights and established the multilinear version of Sawyer's result; however the method do not work without $RH_{\vec{p}}$.
- In fact, we also found that Li, Xue and Yan (2012) introduced a kind of monotone property and established the multilinear version of Sawyer's result.
- More information on the multilinear version of Sawyer's result was in Li and Sun(2016) and Hänninen, Hytönen and Li(2016).

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Reverse Hölder's condition

- Note that if $v = \prod_{i=1}^{m} \omega_i^{p/p_i}$, then the condition $(v, \vec{\omega}) \in A_{\vec{p}}$ implies the reverse Hölder's condition $\vec{\omega} \in RH_{\vec{p}}$ (see Cao and Xue(2016) or Cruz-uribe and Moen(2017)).
- Moreover, the condition $RH_{\vec{p}}$ was also used in Sehba(2017).
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Multilinear weights $B_{\vec{p}}$ and $A_{\vec{p}} - W_{\vec{p}}^{\infty}$ and weighted inequalities

In addition, Chen and Damián (2013) investigated a bound $B_{\vec{p}}$ and a mixed bound $A_{\vec{p}} - W_{\vec{p}}^{\infty}$ for the multilinear maximal operator, which are the multilinear versions of Hytönen-Pérez type weighted estimates (see Hytönen and Pérez(2013)).

Maximal function on the upper half-space $\mathbb{R}^{n+1}_+ = \{(x,t) : x \in \mathbb{R}^n, t \geq 0\}$

Given a function f on $\mathbb{R}^n$, we define a maximal function on the upper half-space $\mathbb{R}^{n+1}_+ = \{(x,t) : x \in \mathbb{R}^n, t \geq 0\}$ by setting

$$\widetilde{M}f(x,t) = \sup_{x \in Q,\, l(Q) \geq t} \frac{1}{|Q|} \int_Q |f(x)| dx,$$

where Q is a cube with its sides parallel to the coordinate axes and $|Q|$ is the Lebesgue measure of Q.

Poisson integral

The maximal function controls the Poisson integral

$$Pf(x,t) = \int_{\mathbb{R}^n} f(y)P(x-y,t)dy \quad x \in \mathbb{R}^n,\ t \geq 0$$

where

$$P(x,t) = c_n \frac{t}{(|x|^2 + t^2)^{\frac{n+1}{2}}}$$

is the Poisson Kernel.

- Let μ be a measure on $\mathbb{R}^{n+1}_+$ and v a weight on $\mathbb{R}^n$.
- Carleson (1962) characterized the positive Borel measures μ on $\mathbb{R}^{n+1}_+$ such that $\widetilde{M}$ is of strong type (p,p) for $p>1$ and of weak type $(1,1)$.
- Later on, Fefferman and Stein (1971) found a condition on the pair (μ, v) to be sufficient for the boundedness of the maximal operator $\widetilde{M}$ from $L^p(\mathbb{R}^n, v)$ into $L^p(\mathbb{R}^{n+1}_+, \mu)$ for $p>1$ and from $L^1(\mathbb{R}^n, v)$ into $L^{1,\infty}(\mathbb{R}^{n+1}_+, \mu)$.

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- Let $p>1$, Ruiz (1985) and Ruiz and Torrea (1985) obtained the exact conditions on the pair (μ, v) for maximal operator $\widetilde{M}$ to be a bounded operator from $L^p(\mathbb{R}^n, v)$ into $L^{p,\infty}(\mathbb{R}^{n+1}_+, \mu)$ and from $L^p(\mathbb{R}^n, v)$ into $L^p(\mathbb{R}^{n+1}_+, \mu)$, respectively.
- Recently, Rivera-Ríos (2016) studied quantitative versions of weighted estimates obtained by Ruiz (1985) and Ruiz and Torrea (1985).

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- Recently, Rivera-Ríos (2016) studied quantitative versions of weighted estimates obtained by Ruiz (1985) and Ruiz and Torrea (1985).

The aim of this paper is to give some multilinear analogues of the above mentioned results for the maximal function on the upper half-space $\mathbb{R}^{n+1}_{+}$.

- Given $\overrightarrow{f} = (f_1, \ldots, f_m)$, we define the multilinear maximal operator $\mathfrak{M}$ on the upper half-space $\mathbb{R}_+^{n+1}$ by

$$\mathfrak{M}(\overrightarrow{f})(x,t) = \sup_{x \in Q,\, l(Q) \geq t} \prod_{i=1}^{m} \frac{1}{|Q|} \int_Q |f_i(y_i)| dy_i$$

where Q is a cube with its sides parallel to the coordinate axes and $|Q|$ is the Lebesgue measure of Q.

- We provide some weighted estimates for $\mathfrak{M}$. Our approaches are mainly based on a Calderón-Zygmund type decomposition suited to the multilinear setting and the multilinear Carleson embedding theorem.

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- We provide some weighted estimates for $\mathfrak{M}$. Our approaches are mainly based on a Calderón-Zygmund type decomposition suited to the multilinear setting and the multilinear Carleson embedding theorem.

Preliminaries and definitions

Let Q be a cube in $\mathbb{R}^n$. We denote by $\tilde{Q}$ the cube built as follows

$$\widetilde{Q} = \left\{(x,t) \in \mathbb{R}^{n+1}_+ \,:\, x \in Q, \text{ and } 0 \leq t < l(Q)\right\},$$

in other words, $\widetilde{Q}$ is the cube in $\mathbb{R}^{n+1}_+$ having Q as a face.

Definition 2.1

Let $\overrightarrow{P} = (p_1, \cdots, p_m)$ *with* $1 < p_1, \cdots, p_m < \infty$ *and* $1/p_1 + \cdots + 1/p_m = 1/p$. *Given* $\overrightarrow{\omega} = (\omega_1, \cdots, \omega_m)$, *set*

$$v_{\overrightarrow{\omega}} = \prod_{i=1}^{m} \omega_i^{p/p_i}.$$

We say that $\overrightarrow{\omega}$ *satisfies the multilinear* $A_{\overrightarrow{P}}$ *condition if*

$$[\overrightarrow{\omega}]_{A_{\overrightarrow{P}}} := \sup_Q \left(\frac{1}{|Q|} \int_Q v_{\overrightarrow{w}}(x)dx \right)^{1/p} \prod_{i=1}^{m} \left(\frac{1}{|Q|} \int_Q \omega_i^{1-p_i'}(x)dx \right)^{1/p_i'} < \infty,$$

where the supremum is taken over all cubes in $\mathbb{R}^n$ *and* $[\overrightarrow{\omega}]_{A_{\overrightarrow{P}}}$ *is called the* $A_{\overrightarrow{P}}$ *constant of* $\overrightarrow{\omega}$.

Definition 2.2

Let $\overrightarrow{P} = (p_1, \cdots, p_m)$ with $1 < p_1, \cdots, p_m < \infty$ and $1/p_1 + \cdots + 1/p_m = 1/p$. Let μ be a Borel measure on $\mathbb{R}^{n+1}_+$. We denote $\overrightarrow{w} = (w_1, \cdots, w_m)$, where w_i, $i = 1, 2, \cdots, m$ be weights in $\mathbb{R}^n$. We say that $(\mu, \overrightarrow{w})$ satisfies the multilinear $A'_{\overrightarrow{P}}$ condition if

$$[\mu, \overrightarrow{w}]_{A'_{\overrightarrow{P}}} := \sup_Q \Big(\frac{\mu(\widetilde{Q})}{|Q|}\Big)^{1/p} \prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q w_i^{1-p'_i}\,dx\right)^{1/p'_i} < \infty,$$

where the supremum is taken over all cubes in $\mathbb{R}^n$ and $[\mu, \overrightarrow{w}]_{A'_{\overrightarrow{P}}}$ is called the $A'_{\overrightarrow{P}}$ constant of $(\mu, \overrightarrow{w})$.

Definition 2.3

Let μ be a Borel measure on $\mathbb{R}_+^{n+1}$. Let v be a weight in $\mathbb{R}^n$. We say that (μ, v) satisfies the C_∞ condition if

$$[\mu, v]_{C_\infty} := \sup_Q \mu(\widetilde{Q})\Big(\int_Q v(x)dx\Big)^{-1} < \infty,$$

where the supremum is taken over all cubes in $\mathbb{R}^n$ and $[\mu, v]_{C_\infty}$ is called the C_∞ constant of (μ, v).

Definition 2.4

Let μ be a Borel measure on $\mathbb{R}^{n+1}_+$. Let v be a weight in $\mathbb{R}^n$. We say that (μ, v) satisfies the C_0 condition if

$$[\mu, v]_{C_0} := \sup_{Q} \mu(\widetilde{Q})^{-1}\Big(\int_Q v(x)dx\Big) < \infty,$$

where the supremum is taken over all cubes in $\mathbb{R}^n$ and $[\mu, v]_{C_0}$ is called the C_0 constant of (μ, v).

Definition 2.5

Let $\overrightarrow{P} = (p_1, \cdots, p_m)$ *with* $1 < p_1, \cdots, p_m < \infty$ *and* $1/p_1 + \cdots + 1/p_m = 1/p$. *Let* μ *be a Borel measure on* $\mathbb{R}^{n+1}_+$. *We denote* $\overrightarrow{w} = (w_1, \cdots, w_m)$, *where* ω_i, $i = 1, 2, \cdots, m$ *be weights in* $\mathbb{R}^n$. *We say that* $(\mu, \overrightarrow{\omega})$ *satisfies the multilinear* $S'_{\overrightarrow{P}}$ *condition if*

$$[\mu, \overrightarrow{\omega}]_{S'_{\overrightarrow{p}}} \triangleq \sup_Q \Big(\int_{\widetilde{Q}} \mathfrak{M}(\overrightarrow{\sigma\chi_Q})^p d\mu \Big)^{\frac{1}{p}} \Big(\prod_{i=1}^m \sigma_i(Q)^{\frac{1}{p_i}} \Big)^{-1} < \infty,$$

where $\overrightarrow{\sigma\chi_Q} = (\omega_1^{1-p_1'}\chi_Q, \cdots, \omega_m^{1-p_m'}\chi_Q)$, *the supremum is taken over all cubes in* $\mathbb{R}^n$ *and* $[\mu, \overrightarrow{w}]_{S'_{\overrightarrow{P}}}$ *is called the* $S'_{\overrightarrow{P}}$ *constant of* $(\mu, \overrightarrow{w})$.

Definition 2.6

Let $\overrightarrow{P} = (p_1, \cdots, p_m)$ *with* $1 < p_1, \cdots, p_m < \infty$ *and* $1/p_1 + \cdots + 1/p_m = 1/p$. *Let* μ *be a Borel measure on* $\mathbb{R}^{n+1}_+$. *We denote* $\overrightarrow{w} = (w_1, \cdots, w_m)$, *where* ω_i, $i = 1, 2, \cdots, m$ *be weights in* $\mathbb{R}^n$. *We say that* $(\mu, \overrightarrow{w})$ *satisfies the* $B'_{\overrightarrow{P}}$ *condition if*

$$[\mu, \overrightarrow{w}]_{B'_{\overrightarrow{P}}} := \sup_Q \Big(\frac{\mu(\widetilde{Q})}{|Q|}\Big)^{\frac{1}{p}} \prod_{i=1}^m \frac{w_i(Q)}{|Q|} \exp\Big(\frac{1}{|Q|}\int_Q \log \prod_{i=1}^m w_i^{-\frac{1}{p_i}} dx\Big) < \infty.$$

where the supremum is taken over all cubes in $\mathbb{R}^n$.

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Let $\overrightarrow{P} = (p_1, \cdots, p_m)$ *with* $1 < p_1, \cdots, p_m < \infty$ *and* $1/p_1 + \cdots + 1/p_m = 1/p$. *We denote* $\overrightarrow{w} = (w_1, \cdots, w_m)$, *where* ω_i, $i = 1, 2, \cdots, m$ *be weights in* $\mathbb{R}^n$. *We say that* $\overrightarrow{\omega}$ *satisfies the reverse Hölder's condition* $RH_{\overrightarrow{p}}$, *if*

$$\prod_{i=1}^{m} \Big(\int_Q \sigma_i dx \Big)^{\frac{p}{p_i}} \leq C \int_Q \prod_{i=1}^{m} \sigma_i^{\frac{p}{p_i}} dx, \ \forall \text{ cube } Q,$$

where $\sigma_i = \omega_i^{1-p_i'}$, $i = 1, \cdot \cdot \cdot, m$ *and the smallest constant* C *is denoted by* $[\overrightarrow{\omega}]_{RH_{\overrightarrow{p}}}$.

Definition 2.8

Let $\vec{P} = (p_1, \cdots, p_m)$ *with* $1 < p_1, \cdots, p_m < \infty$ *and* $1/p_1 + \cdots + 1/p_m = 1/p$. *We denote* $\vec{w} = (w_1, \cdots, w_m)$, *where* ω_i, $i = 1, 2, \cdots, m$ *be weights in* $\mathbb{R}^n$. *We say that* $\vec{w}$ *satisfies the* $W^{\infty}_{\vec{P}}$ *condition if*

$$[\vec{w}]_{W^{\infty}_{\vec{P}}} = \sup_Q \Big(\int_Q \prod_{i=1}^{m} M(w_i \chi_Q)^{\frac{p}{p_i}} dx \Big) \Big(\int_Q \prod_{i=1}^{m} w_i^{\frac{p}{p_i}} dx \Big)^{-1} < \infty,$$

where the supremum is taken over all cubes in $\mathbb{R}^n$.

Theorem 2.1

Let $\overrightarrow{P} = (p_1, \cdots, p_m)$ *with* $1 < p_1, \cdots, p_m < \infty$ *and* $1/p_1 + \cdots + 1/p_m = 1/p$. *Let* μ *be a Borel measure on* $\mathbb{R}^{n+1}_+$. *Let* $\omega_1, \cdots, \omega_m$ *be weights in* $\mathbb{R}^n$. *Then the following statements are equivalent:*

(1) $(\mu, \overrightarrow{w})$ *satisfies the multilinear* $A'_{\overrightarrow{P}}$ *condition;*

(2) *There exists a positive constant* C *such that*

$$\lambda\mu\left(\{(x,t) \in \mathbb{R}^{n+1}_+ : \mathfrak{M}(\overrightarrow{f}) > \lambda\}\right)^{\frac{1}{p}} \leq C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\mathbb{R}^n, \omega_i)},$$

for any $\overrightarrow{f} \in \prod_{i=1}^{m} L^{p_i}(\mathbb{R}^n, \omega_i)$ *and* $\lambda > 0$.

Moreover, if we denote the smallest constants C in (2) by $\|\mathfrak{M}\|$, then we have

$$[v, \overrightarrow{\omega}]_{A'_{\overrightarrow{p}}} \leq \|\mathfrak{M}\| \lesssim [v, \overrightarrow{\omega}]'_{A_{\overrightarrow{p}}}.$$

Theorem 2.2

Suppose $1 < p_1, \ldots, p_m < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$, $\vec{w} \in A'_{\vec{P}}$ *and* $(\mu, v_{\vec{w}}) \in C_0$. *Then*

$$\|\mathfrak{M}(\vec{f})\|_{L^p(\mathbb{R}^{n+1}_+,\mu)} \lesssim [\mu, v_{\vec{w}}]_{C_0}^{1/p} [\vec{w}]_{A'_{\vec{P}}}^{\bar{p}} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\mathbb{R}^n, w_i)}, \qquad (2.1)$$

where $\bar{p} = \max\{p'_1, \cdots, p'_m\}$.

Theorem 2.3

Suppose $1 < p_1, \ldots, p_m < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$, $\vec{w} \in A_{\vec{P}}$ *and* $(\mu, v_{\vec{w}}) \in C_\infty$. *Then*

$$\|\mathfrak{M}(\vec{f})\|_{L^p(\mathbb{R}^{n+1}_+,\mu)} \lesssim [\mu, v_{\vec{w}}]^{1/p}_{C_\infty}[\vec{w}]^{\bar{p}}_{A_{\vec{P}}} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\mathbb{R}^n,w_i)},$$

where $\bar{p} = \max\{p_1', \cdots, p_m'\}$.

Theorem 2.4

Suppose $1 < p_1, \ldots, p_m < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$. *If* $(v, \vec{\omega}) \in B'_{\vec{p}}$, *then the following statements are valid:*

(1) *There exists a positive constant* C *such that*

$$\|\mathfrak{M}(\vec{f})\|_{L^p(\mathbb{R}^{n+1}_+,\mu)} \leq C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\mathbb{R}^n,\omega_i)}, \ \forall f_i \in L^{p_i}(\mathbb{R}^n, \omega_i);$$

(2) *There exists a positive constant* C *such that*

$$\|\mathfrak{M}(\vec{f\sigma})\|_{L^p(\mathbb{R}^{n+1}_+,\mu)} \leq C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\mathbb{R}^n,\sigma_i)}, \ \forall f_i \in L^{p_i}(\mathbb{R}^n, \sigma_i).$$

Moreover, we denote the smallest constants C in (1) and (2) by $\|\mathfrak{M}\|$ and $\|\mathfrak{M}\|'$, respectively. Then it follows that

$$\|\mathfrak{M}\| = \|\mathfrak{M}\|' \lesssim [\mu, \vec{\omega}]_{B'_{\vec{p}}}.$$

Theorem 2.5

Suppose $1 < p_1, \ldots, p_m < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$. *If* $(\mu, \vec{\omega}) \in A'_{\vec{p}}$ *and* $\vec{\omega} \in W^{\infty}_{\vec{p}}$, *then the following statements are valid:*

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Moreover, we denote the smallest constants C in (1) and (2) by $\|\mathfrak{M}\|$ and $\|\mathfrak{M}\|'$, respectively. Then it follows that $\|\mathfrak{M}\| = \|\mathfrak{M}\|' \lesssim [\mu, \overrightarrow{\omega}]_{A'_{\overrightarrow{p}}} [\overrightarrow{\omega}]_{W^{\infty}_{\overrightarrow{p}}}^{\frac{1}{p}}$.

Theorem 2.6

Suppose $1 < p_1, \ldots, p_m < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$. *If* $(\omega_1, \omega_2, \cdots, \omega_m) \in RH_{\vec{p}}$, *then the following statements are equivalent:*

(1) *There exists a positive constant* C *such that*

$$\|\mathfrak{M}(\vec{f})\|_{L^p(\mathbb{R}^{n+1}_+,\mu)} \leq C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\mathbb{R}^n,\omega_i)}, \quad \forall f_i \in L^{p_i}(\mathbb{R}^n, \omega_i);$$

(2) $(\mu, \vec{\omega})$ *satisfies the condition* $S_{\vec{p}}$.

Moreover, we denote the smallest constants C *in (1) by* $\|\mathfrak{M}\|$. *Then it follows that*

$$[v, \vec{\omega}]_{S_{\vec{p}}} \leq \|\mathfrak{M}\| \lesssim [\vec{\omega}]^{\frac{1}{p}}_{RH_{\vec{p}}} [v, \vec{\omega}]_{S_{\vec{p}}}.$$

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