Hardy-type inequalities for fractional powers of the sublaplacian on the Heisenberg group

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(Based on joint work with Luz Roncal)

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Classical Hardy inequality

Due to G. H. Hardy. . .

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...but also contributions from E. Landau, G. Pólya, I. Schur, and M. Riesz (cf. A. Kufner, L. Maligranda, and L-E. Persson, *The prehistory of the Hardy inequality*, Amer. Math. Monthly) **113** (2006), 715–732

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A statement of Hardy inequality (Hardy 1920, 1925, also Hardy-Littlewood-Pólya, *Inequalities*):

Given f nonnegative (measurable) on $(0, \infty)$, then

$$\int_0^\infty \left| \frac{1}{x} \int_0^x f(t)\, dt \right|^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^\infty |f(x)|^p \, dx$$

when $p > 1$ and RHS is finite

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Given f nonnegative (measurable) on $(0,\infty)$, let $F(x)=\int_0^x f(t)\,dt$, then

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Higher dimension $d \geq 3$, $p = 2$

$$0 \leq \int_{\mathbb{R}^d} \left| \nabla f + \alpha \frac{x}{|x|^2} f \right|^2 = \int_{\mathbb{R}^d} |\nabla f|^2 + \left(\alpha^2 - (d-2)\alpha \right) \int_{\mathbb{R}^d} \left| \frac{f(x)}{x} \right|^2$$

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and for all $\nu > \frac{(d-2)^2}{4}$ then $\Delta - \frac{\nu}{|x|^2}$ is not bounded below

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2. A maximal average of a function f is "dominated" in L^p sense by f:
   - Generalisation of Hardy's inequality, seen as maximal inequalities for which Hardy–Littlewood maximal inequality is the model

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Taken from `http://mathoverflow.net/questions/48292/applications-of-hardys-inequality?`

Fractional version of Hardy inequality

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$$\int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^{2s}}\, dx \leq C_{d,s} \langle \Delta^s f, f \rangle, \qquad C_{d,s} = 4^{-s} \frac{\Gamma(\frac{d-2s}{4})^2}{\Gamma(\frac{d+2s}{4})^2}$$

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- We may consider also a (weaker) Hardy with a non-homogeneous potential

$$\int_{\mathbb{R}^n} \frac{|f(x)|^2}{(1+|x|^2)^s}\,dx \leq b_{d,s} \langle \Delta^s f, f \rangle$$

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- Recall: for $0 < s < d/2$, $f \in C_0^\infty(\mathbb{R}^d)$

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  2. *Ground-state representation*. Define

$$h_s[f] := \int_{\mathbb{R}^d} |\xi|^{2s} |\widehat{f}|^2\, d\xi - C_{d,s} \int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^{2s}}\, dx$$

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then, calling $g(x) = f(x)|x|^{\frac{d-2s}{2}}$, for $0 < s < \min\{1, d/2\}$ we have

$$h_s[f] = a_{d,s}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\big(g(x)-g(y)\big)^2}{|x-y|^{d+2s}}\,\frac{dx}{|x|^{\frac{d-2s}{2}}}\,\frac{dy}{|y|^{\frac{d-2s}{2}}} \ge 0 \to \text{recover Hardy's ineq.}$$

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$$A = \int_0^\infty \lambda dE(\lambda) \Leftrightarrow \langle Af, g\rangle = \int_0^\infty \lambda dE_{f,g}(\lambda), \quad f \in \mathrm{Dom}\, A, \;\; g \in L^2(X)$$

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- **Aim:** Prove a Hardy-type inequality of the form

$$C_s \int_X h_s(x)|f(x)|^2\, d\eta(x) \le \langle A_s f, f\rangle$$

with h_s an appropriate positive function, and $C_s > 0$ sharp

Generalisation of the method - two steps

1. **Integral representation and an expression for $\langle A_s f, f\rangle$.** Let us assume

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$$\mathcal{H}_s[f] := \langle A_s f, f\rangle - C_s \int_X \frac{\widetilde{w}_s(x)}{w_s(x)} |f(x)|^2\, d\eta(x), \quad \widetilde{w}_s(x) := A_s w_s(x)$$

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Let w_s be a positive function and set $g(x) = f(x)(w_s(x))^{-1}$. Then

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  - w_s is in most cases related to the fundamental solution of A_s
  - inequality above is not necessarily a sharp Hardy inequality, but an intermediate step towards a sharp Hardy inequality $\rightarrow$ minimize $C_s \frac{\widetilde{w}_s(x)}{w_s(x)}$

Application of the method

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- We will **not** prove Hardy-type inequalities for the *pure* fractional powers of the above operators, but for the **conformally invariant fractional powers**

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2. Laguerre operator, harmonic oscillator, Dunkl–Hermite
3. Euclidean Laplacian $\to$ revisit

- We will **not** prove Hardy-type inequalities for the *pure* fractional powers of the above operators, but for the **conformally invariant fractional powers**
- Definition of these **conformally invariant fractional powers** via spectral theorem

Application of the method

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- Step 2 in FLS-method: non-negativity of ground state representation $\rightarrow$ generalisation of Cowling-Haagerup formula

Hardy-type inequality on the Heisenberg group

The Heisenberg group $\mathbb{H}^d$: preliminaries

- $\mathbb{H}^d$ is the Lie group with underlying manifold $\mathbb{C}^d \times \mathbb{R}$ and multiplication

$$(z, w)(z', w') = \left(z + z', w + w' + \frac{1}{2}\operatorname{Im}(z \cdot \bar{z'})\right)$$

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$$X_j = \left(\frac{\partial}{\partial x_j} + \frac{1}{2} y_j \frac{\partial}{\partial w}\right), \quad Y_j = \left(\frac{\partial}{\partial y_j} - \frac{1}{2} x_j \frac{\partial}{\partial w}\right), \quad T = \frac{\partial}{\partial w}, \quad j = 1, 2, \ldots, d$$

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- The sublaplacian is

$$\mathcal{L} := -\sum_j (X_j^2 + Y_j^2)$$

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whenever $f \in L^2(\mathbb{H}^d)$ and polarizing, we get Parseval formula

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- The convolution on $\mathbb{H}^d$

$$f * g(x) = \int_{\mathbb{H}^d} f(xy^{-1})g(y)dy, \quad x, y \in \mathbb{H}^d$$

and also the λ-twisted convolution $(f*g)^\lambda(z) = f^\lambda *_\lambda g^\lambda$, $z \in \mathbb{C}^d$

Spectral theory of the sublaplacian

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Here $\varphi_k^\lambda(z) = L_k^{n-1}(\frac{1}{2}|\lambda||z|^2)e^{-\frac{1}{4}|\lambda||z|^2}$ are the Laguerre functions of type $(n-1)$

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Important: (i) The Laguerre and Hermite functions are related via *Weyl transform*

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2. Taking the Weyl transform and using the relation above

   $$\widehat{f}(\lambda) = \sum_k c_k^\lambda(f^\lambda)P_k(\lambda),$$

3. The inversion formula for the FT on $\mathbb{H}^d$ can be put in the form

   $$f(z,w) = c_d \int_{-\infty}^{\infty} |\lambda|^d \Big(\sum_k f^\lambda *_\lambda \varphi_k^\lambda(z)\Big) e^{-i\lambda w} d\lambda$$

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- Then, a natural way to define fractional powers of $\mathcal{L}$ is
$$\mathcal{L}^s f(z,w) = c_d \int_{-\infty}^{\infty} |\lambda|^d \Big(\sum_k ((2k+d)|\lambda|)^s f^\lambda *_\lambda \varphi_k^\lambda(z)\Big) e^{-i\lambda w}\, d\lambda$$

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Fractional powers of the sublaplacian

- The above means that $\mathcal{L}_s$ is the operator

$$\mathcal{L}_s := (2|T|)^s \frac{\Gamma\left(\frac{\mathcal{L}}{2|T|} + \frac{1+s}{2}\right)}{\Gamma\left(\frac{\mathcal{L}}{2|T|} + \frac{1-s}{2}\right)}$$

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- Thus $\mathcal{L}_s$ corresponds to the Fourier multiplier

$$(2|\lambda|)^s \frac{\Gamma\left(\frac{2k+d}{2} + \frac{1+s}{2}\right)}{\Gamma\left(\frac{2k+d}{2} + \frac{1-s}{2}\right)}, \quad k \in \mathbb{N}$$

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3. Relation: $\mathcal{L}_s = U_s \mathcal{L}^s$, U_s bounded on $L^2(\mathbb{H}^d)$ (Stirling's formula)
4. $\mathcal{L}_s$ is not that weird: it is conformally invariant (see e.g. T. P. Branson, L. Fontana, and C. Morpurgo (2013))

The approach through heat semigroup to define $\mathcal{L}_s$

- We have

$$e^{-t\mathcal{L}} f = f * q_t,$$

where

$$\int_{-\infty}^{\infty} q_t(z,w) e^{i\lambda w}\, dw = c_d |\lambda|^d \sum_{k=0}^{\infty} e^{-(2k+n)|\lambda| t} \varphi_k^\lambda(z) =: q_t^\lambda(z)$$

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$$\int_{-\infty}^{\infty} \mathcal{K}_t^s(z,w)e^{i\lambda w}\,dw = q_t^\lambda(z)\Big(\frac{\lambda t}{\sinh \lambda t}\Big)^{s+1}$$

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to obtain

$$c_d |\lambda|^n \sum_k (2|\lambda|)^s \frac{\Gamma\left(\frac{2k+d}{2}+\frac{1+s}{2}\right)}{\Gamma\left(\frac{2k+d}{2}+\frac{1-s}{2}\right)} f^\lambda *_\lambda \varphi_k^\lambda(z)$$

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Consequently

$$\mathcal{L}_s f(z,w) = C_s \int_0^\infty \left(f(z,w) - f * \mathcal{K}_t^s(z,w) \right) t^{-s-1}\, dt$$

The approach through heat semigroup to define $\mathcal{L}_s$

$x = (z, w)$. We use $\int_{\mathbb{H}^d} \mathcal{K}_t^s(x)\, dx = 1$, so

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Theorem (Integral representation for $\mathcal{L}_s$)

$$\mathcal{L}_s f(x) = \frac{1}{|\Gamma(-s)|} \int_{\mathbb{H}^d} \big(f(x) - f(y)\big)\mathcal{K}_s(y^{-1}x)\, dy, \quad f \in C_0^\infty(\mathbb{H}^d)$$

where $\mathcal{K}_s(z, w) = c_{d,s}|(z, w)|^{-Q-2s}$ *(*$Q = 2d + 2$*), and* $c_{d,s} > 0$ *is explicit*

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RHS: A bit of algebra: for any $f \in W^{s,2}(\mathbb{H}^d)$, **RHS** above equals

$$a_{d,s} \int_{\mathbb{H}^d} \int_{\mathbb{H}^d} \left(|f(x) - f(y)|^2 - \left| \frac{f(x)}{G(x)} - \frac{f(y)}{G(y)} \right|^2 G(x) G(y) \right) \frac{dx \, dy}{|y^{-1}x|^{Q+2s}}$$

$$= \langle \mathcal{L}_s f, f \rangle - \text{non-negative quantity}$$

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LHS above can be simplified to yield

$$\langle \mathcal{L}_s F, G \rangle = C_{\delta,Q,s} \int_{\mathbb{H}^d} \frac{|f(x)|^2}{\left((\delta + \frac{1}{4}|z|^2)^2 + w^2\right)^s}\, dx$$

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Summarizing, we have proved that:

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Since

$$\mathcal{H}_s[f] := \langle \mathcal{L}_s f, f\rangle - C_{\delta,Q,s}\int_{\mathbb{H}^d}\frac{|f(z,w)|^2}{\left((\delta+\frac{1}{4}|z|^2)^2+w^2\right)^s}dz\,dw$$

we get

$$\mathcal{H}_s[f] \geq 0$$

The key fact in the proof of the result on $\mathcal{H}_s[F]$

- Recall $u_{s,\delta}(z,w) = \left((\delta + \frac{1}{4}|z|^2)^2 + w^2\right)^{-\frac{s+d+1}{2}}$, $G(x) = u_{-s,\delta}$

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A Hardy-type inequality for $\mathcal{L}_s$

From the latter result:

Theorem (Hardy inequality with non-homogeneous weight)

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  - we do not know whether the inequality is optimal!
  - the procedure is the same: integral representations for Λ_s and $\langle \Lambda_s f, f \rangle$ + ground state representation

Heisenberg uncertainty inequalities

They can be obtained from Hardy ineq. (see N. Garofalo and E. Lanconelli (1990) for $\mathcal{L}$)

Corollary (Uncertainty principles for $\mathcal{L}_s$ (and Λ_s))

$$(4\delta)^s \frac{\Gamma\big(\frac{1+d+s}{2}\big)^2}{\Gamma\big(\frac{1+d-s}{2}\big)^2}\Big(\int_{\mathbb{H}^d} |f(z,w)|^2\, dz\, dw\Big)^2$$

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Proof. Let $\mathcal{W}(z,w)$ be a (homog. or non-homog.) weight; by Cauchy–Schwarz

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The last integral is bounded by $\langle \mathcal{L}_s f, f\rangle^{1/2}$ or $\langle \Lambda_s f, f\rangle^{1/2}$ times the corresponding constant, by Hardy's inequality

Consistency of the method:
Hardy-type inequality for the Laguerre operator

The fractional powers of the Laguerre operator

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- For $0 \le s \le 1$ we define $L_{\alpha,s}$ by

$$L_{\alpha,s}f(r) = \sum_k 4^s \frac{\Gamma\left(\frac{4k+2\alpha+2}{4} + \frac{1+s}{2}\right)}{\Gamma\left(\frac{4k+2\alpha+2}{4} + \frac{1-s}{2}\right)} f *_\alpha \varphi_k^\alpha(r).$$

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Remark: Laguerre setting can be seen as a particular case of Heisenberg (functions which are radial in the first variable)$\to$ hints in the reasoning on $\mathbb{H}^n$

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Again the key point is to compute the action of $L_{\alpha,s}$ on $w_{\alpha,-s}(r)$

$$\int_0^\infty L_{\alpha,s} f(r) w_{\alpha,-s}(r) \, d\mu_\alpha(r) = (4\delta)^s \frac{\Gamma(\frac{\alpha+2+s}{2})^2}{\Gamma(\frac{\alpha+2-s}{2})^2} \int_0^\infty f(r) w_{\alpha,s}(r) \, d\mu_\alpha(r)$$

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- Hardy inequalities with an homogeneous weight could be stated

Thanks for your attention