Operator-valued local Hardy spaces and applications to pseudo-differential operators

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Background

The mapping properties of Pseudo-differential operators (abbreviated as ψDO's in the sequel) are widely studied in the field of Harmonic analysis, for example, their boundedness on L_p, $1 < p < \infty$, Hardy and Triebel-Lizorkin spaces.
In this talk, we will extend some of the classical results mentioned above to the noncommutative setting.

Local Hardy spaces

Motivation to study local Hardy spaces: Some good pseudo-differential operators are not bounded on $H_1(\mathbb{R}^d)$. For example, the multiplication by any Schwartz function is a pseudo-differential operator, however, it does not send $H_1(\mathbb{R}^d)$ to $H_1(\mathbb{R}^d)$.

Let Γ be the truncated cone $\left\{(t,\varepsilon) \in \mathbb{R}_+^{d+1} : |t| < \varepsilon < 1\right\}$, define

$$s(f)(s) = \left(\int_\Gamma |\nabla (P_\varepsilon * f)(s+t,\varepsilon)|^2 \frac{dyd\varepsilon}{\varepsilon^{d-1}} \right)^{\frac{1}{2}}$$

where P is the Poisson kernel on $\mathbb{R}^d$ and $P_\varepsilon(s) = \frac{1}{\varepsilon^d} P(\frac{s}{\varepsilon})$.

- **D. Goldberg (1979)**: If $p > (d-1)/d$, define the local Hardy space h_p as following:

$$h_p(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{h_p} < \infty\},$$

with $\|f\|_{h_p} = \|P * f\|_{L_p} + \|s(f)\|_{L_p}$.

Moving to the noncommutative setting:

- Motivated by Goldberg, we want to define the operator-valued local Hardy spaces, and also get the boundedness of Pseudo-differential operators on them.
- Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal semifinite faithful trace τ.
  For $1 \leq p \leq \infty$, let $L_p(\mathcal{M})$ be the noncommutative L_p-space associated to $(\mathcal{M}, \tau)$. The norm of $L_p(\mathcal{M})$, $1 \leq p < \infty$ is given by

  $$\|x\|_p = \tau(|x|^p)^{\frac{1}{p}} \text{ with } |x| = (x^*x)^{1/2}.$$

  Set $L_\infty(\mathcal{M}) = \mathcal{M}$.

Lusin area square function and Littlewood-Paley g-function

Operator-valued Hardy spaces

$$S^c(f)(s) = \left(\int_\Gamma \left| \frac{\partial}{\partial \varepsilon} P_\varepsilon(f)(s+t) \right|^2 \frac{dt d\varepsilon}{\varepsilon^{d-1}} \right)^{\frac{1}{2}}$$

$$S^r(f)(s) = \left(\int_\Gamma \left| \frac{\partial}{\partial \varepsilon} P_\varepsilon(f^*)(s+t) \right|^2 \frac{dt d\varepsilon}{\varepsilon^{d-1}} \right)^{\frac{1}{2}}$$

$$\left(\Gamma = \left\{ (t, \varepsilon) \in \mathbb{R}_+^{d+1} : |t| < \varepsilon \right\} \right)$$

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$$g^c(f)(s) = \left(\int_0^1 \varepsilon \left| \frac{\partial}{\partial \varepsilon} P_\varepsilon(f)(s) \right|^2 d\varepsilon \right)^{\frac{1}{2}}$$

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$$\|f\|_{H_p^c} = \|S^c(f)\|_{L_p}$$

Operator-valued local Hardy spaces

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$$\|f\|_{h_p^c} = \|s^c(f)\|_{L_p} + \|P * f\|_{L_p}$$

where $P_\varepsilon(f)$ denotes $P_\varepsilon * f$.

Operator-valued local Hardy spaces

Let $f \in \mathcal{S}'(\mathbb{R}^d; L_\infty(\mathcal{M}) + L_1(\mathcal{M}))$, for $1 \leq p < \infty$, we define

$$\|f\|_{h_p^c(\mathbb{R}^d,\mathcal{M})} = \|s^c(f)\|_{L_p(L_\infty(\mathbb{R}^d)\overline{\otimes}\mathcal{M})} + \|P * f\|_{L_p(L_\infty(\mathbb{R}^d)\overline{\otimes}\mathcal{M})},$$

the **column local Hardy spaces** are defined by

$$h_p^c\left(\mathbb{R}^d, \mathcal{M}\right) = \left\{f : \|f\|_{h_p^c} < \infty\right\},$$

and the **row local Hardy spaces** are defined by

$$h_p^r\left(\mathbb{R}^d, \mathcal{M}\right) = \left\{f : \|f^*\|_{h_p^c} < \infty\right\},$$

equipped with the norm $\|f\|_{h_p^r} = \|f^*\|_{h_p^c}$.

$$\begin{aligned} h_p\left(\mathbb{R}^d, \mathcal{M}\right) &= h_p^c\left(\mathbb{R}^d, \mathcal{M}\right) + h_p^r\left(\mathbb{R}^d, \mathcal{M}\right) \text{ for } 1 \leq p \leq 2, \\ h_p\left(\mathbb{R}^d, \mathcal{M}\right) &= h_p^c\left(\mathbb{R}^d, \mathcal{M}\right) \cap h_p^r\left(\mathbb{R}^d, \mathcal{M}\right) \text{ for } 2 < p < \infty. \end{aligned}$$

Replacing the poisson kernel

Let $\Phi \in \mathcal{S}$ with $\int \Phi(s)\, ds = 0$. Assume that Φ is nondegenerate in the following sense:

$$\forall \xi \in \mathbb{R}^d \setminus \{0\}, \exists \varepsilon > 0 \text{ such that } \hat{\Phi}(\varepsilon\xi) \neq 0.$$

There exists $\Psi \in \mathcal{S}$ satisfying the same assumptions as Φ such that

$$\int_0^\infty \widehat{\Phi}(\varepsilon\xi)\widehat{\Psi}(\varepsilon\xi)\frac{d\varepsilon}{\varepsilon} = 1 \quad \text{for any} \quad \xi \in \mathbb{R}^d.$$

We can find two functions φ, ϕ such that $\widehat{\varphi}, \widehat{\phi} \in H_2^\sigma(\mathbb{R}^d)$, $\widehat{\varphi}(0) > 0$, $\widehat{\phi}(0) > 0$ and

$$\widehat{\varphi}(\xi)\widehat{\phi}(\xi) = 1 - \int_0^1 \widehat{\Phi}(\varepsilon\xi)\widehat{\Psi}(\varepsilon\xi)\frac{d\varepsilon}{\varepsilon}.$$

Theorem

For $1 \leq p < \infty$, we have

$$\|\varphi * f\|_{L_p} + \|g_\Phi^c(f)\|_{L_p} \approx \|\varphi * f\|_{L_p} + \|s_\Phi^c(f)\|_{L_p} \approx \|f\|_{h_p^c}.$$

Inhomogeneous operator-valued Triebel-Lizorkin spaces on $\mathbb{R}^d$

Definition

Let $1 \leq p < \infty$ and $\alpha \in \mathbb{R}$. The column Triebel-Lizorkin space $F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$ is defined by

$$F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M}) = \{f \in \mathcal{S}'(\mathbb{R}^d; L_1(\mathcal{M}) + L_\infty(\mathcal{M})) : \|f\|_{F_p^{\alpha,c}} < \infty\},$$

where $\|f\|_{F_p^{\alpha,c}} = \|\varphi * f\|_{L_p(\mathcal{M})} + \|(\int_0^1 \varepsilon^{-2\alpha}|\Phi_\varepsilon * f|^2 \frac{d\varepsilon}{\varepsilon})^{\frac{1}{2}}\|_p$.

Remark

$F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$ *is independent of the choice of the function* Φ.

Basic properties

Let J^α be the Bessel potential of order α which is the operator $(Id - (2\pi)^{-2}\Delta)^{\frac{\alpha}{2}}$. Let I^α be the Riesz potential of order α which is the operator $(-(2\pi)^{-2}\Delta)^{\frac{\alpha}{2}}$.

Proposition

Let $1 \leq p < \infty$ and $\alpha \in \mathbb{R}$. Then

1. $F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M}) \subset F_p^{\beta,c}(\mathbb{R}^d, \mathcal{M})$ if $\alpha > \beta$.
2. $F_p^{0,c}(\mathbb{R}^d, \mathcal{M}) \approx h_p^c(\mathbb{R}^d, \mathcal{M})$.
3. For any $\beta \in \mathbb{R}$, both J^β and I^β are isomorphisms between $F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$ and $F_p^{\alpha-\beta,c}(\mathbb{R}^d, \mathcal{M})$. Moreover, $\|f\|_{F_p^{\alpha,c}} \approx \|J^\beta f\|_{F_p^{\alpha-\beta,c}}$ and $\|f\|_{F_p^{\alpha,c}} \approx \|I^\beta f\|_{F_p^{\alpha-\beta,c}}$

Smooth atomic decompositions of Triebel-Lizorkin spaces

Definition

Let $\alpha \in \mathbb{R}$, and let K and L be two positive integers with $K \geq ([\alpha]+1)_+$ and $L \geq \max\{[-\alpha], -1\}$.

1. A function $b \in L_1^c(\mathcal{M}; L_2^c(\mathbb{R}^d))$ is called an $(\alpha, 1)$-atom if
   1. supp $b \subset Q_{0,k}$;
   2. $\tau\left(\int |D^\gamma b(s)|^2\, ds\right)^{\frac{1}{2}} \leq 1$, for $|\gamma| \leq K$.
2. A function $a \in L_1^c(\mathcal{M}; L_2^c(\mathbb{R}^d))$ is said to be a $(\alpha, Q_{\mu,l})$-sub-atom if
   1. supp $a \subset 3Q_{\mu,l}$;
   2. $\tau\left(\int |D^\gamma a(s)|^2\, ds\right)^{\frac{1}{2}} \leq |Q_{\mu,l}|^{\frac{\alpha}{d} - \frac{|\gamma|}{d}}$, for $|\gamma| \leq K$;
   3. $\int_{\mathbb{R}^d} s^\beta a(s)\, ds = 0$, for $|\beta| \leq L$,
3. A function $g \in L_1^c(\mathcal{M}; L_2^c(\mathbb{R}^d))$ is said to be an $(\alpha, Q_{k,m})$-atom if $g(s) = \sum_{(\mu,l)<(k,m)} d_{\mu,l} a_{\mu l}(s)$ for some $k \in \mathbb{N}_0$ and $m \in \mathbb{Z}^d$, where $a_{\mu,l}$ is an $(\alpha, Q_{\mu,l})$-sub-atom and $d_{\mu,l}$ are complex numbers such that $\left(\sum_{(\mu,l)<(k,m)} |d_{\mu,l}|^2\right)^{\frac{1}{2}} \leq |Q_{k,m}|^{-\frac{1}{2}}$.

Theorem (X.-Xiong)

Let $\alpha \in \mathbb{R}$ and let K and L be be two fixed positive integers satisfying the conditions in the previous definition. Then every $f \in F_1^{\alpha,c}\left(\mathbb{R}^d, \mathcal{M}\right)$ can be represented as

$$f = \sum_{j=1}^{\infty} \mu_j b_j + \lambda_j g_j,$$

where b_j are $(\alpha, 1)$-atoms and g_j are (α, Q)-atoms, μ_j, λ_j are complex numbers with

$$\sum_{j=1}^{\infty} (|\mu_j| + |\lambda_j|) < \infty. \qquad (1)$$

Furthermore, the infimum of the sum in (1) with respect to all admissible representations (where K and L are fixed) is an equivalent norm in $F_1^{\alpha,c}\left(\mathbb{R}^d, \mathcal{M}\right)$.

Application: Pseudo-differential operators

Definition

Let $m \in \mathbb{R}$, $0 \leq \rho \leq 1$ and $0 \leq \delta \leq 1$, then $S^m_{\rho,\delta}$ is the collection of all C^∞ functions $\sigma(s,\xi)$ in $\mathbb{R}^d \times \mathbb{R}^d$ with value in $S_{\mathcal{M}}$ such that

$$\|D_s^\gamma D_\xi^\beta \sigma(s,\xi)\|_{\mathcal{M}} \leq C_{\gamma,\beta}(1+|\xi|)^{m+\delta|\gamma|-\rho|\beta|},$$

for $(s,\xi) \in \mathbb{R}^d \times \mathbb{R}^d$.

Definition

Let $\sigma \in S^m_{\rho,\delta}$, the corresponding pseudo-differential operator (ΨDO's) is a mapping $f \mapsto T^c_\sigma(f)$ given by

$$T^c_\sigma(f)(s) = \int_{\mathbb{R}^d} \sigma(s,\xi)\,\widehat{f}(\xi)\,e^{2\pi i s\xi}d\xi.$$

σ is called the **symbol** of T.

ΨDO's have a parallel description in terms of their kernels,

$$T_\sigma^c(f)(s) = \int_{\mathbb{R}^d} K(s, s-t) f(t) dt,$$

where $K(s,t) = \mathcal{F}^{-1}_{\xi \mapsto t}(\sigma(s,\xi))$.

In the sequel, we will only focus on symbols in the class $S^0_{1,\delta}$. **Why?**

- Even in the classical theory, when $0 \leq \rho < 1$, T_σ with $\sigma \in S^m_{\rho,\delta}$ is not necessarily bounded on L_p, $1 < p < \infty$.
- By the lifting property of $F_p^{\alpha,c}$, we know that the Bessel potential J^α maps $F_p^{\alpha,c}$ isomorphically to $F_p^{0,c}$ and $\|f\|_{F_p^{\alpha,c}} \approx \|J^\alpha f\|_{F_p^{0,c}}$. Moreover, we can easily see that for any $\sigma \in S^0_{1,\delta}$, we have $(1+|\xi|^2)^{\frac{\alpha}{2}}\sigma \in S^\alpha_{1,\delta}$. Thus, if we have the result that $\forall\, \sigma \in S^0_{1,\delta}$, T_σ^c is bounded on $F_p^{\alpha,c}$, we can deduce that $\forall\, \sigma \in S^m_{1,\delta}$, T_σ^c is bounded from $F_p^{\alpha,c}$ to $F_p^{\alpha-m,c}$.

The boundedness of pseudo-differential operators on $F_1^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$

Lemma

Let T_σ^c *be a* ΨDO *with symbol in* $S_{1,\delta}^0$ *with* $0 \le \delta \le 1$. *Then* T_σ^c *admits a kernel* $K(s,t)$ *which satisfies*

$$\begin{aligned} \|D_s^\gamma D_t^\beta K(s,t)\|_{\mathcal{M}} &\le C_{\gamma,\beta}|t|^{-d-|\gamma|-|\beta|}, \quad \forall\, t \in \mathbb{R}^d \setminus \{0\} \\ \|D_s^\gamma D_t^\beta K(s,t)\|_{\mathcal{M}} &\le C_{\gamma,\beta}|t|^{-N}, \quad \forall\, N > 0 \textit{ if } |t| > 1. \end{aligned}$$

The boundedness of pseudo-differential operators on $F_1^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$

Lemma

Let T_σ^c be a ΨDO with symbol in $S^0_{1,\delta}$ with $0\le\delta\le 1$. Then T_σ^c admits a kernel $K(s,t)$ which satisfies

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Lemma

Let σ be a symbol of the class $S^0_{1,\delta}$ with $0\le\delta<1$ and T_σ^c the corresponding ΨDO's, then for every sub-atom $a_{\mu l}$ and any $M\in\mathbb{N}_0$, we have

$$\tau\left(\int\left(1+2^\mu\left|s-l\cdot 2^{-\mu}\right|\right)^{d+M}\left|D^\gamma T_\sigma^c a_{\mu l}(s)\right|^2 ds\right)^{\frac{1}{2}}\lesssim|Q_{\mu,l}|^{\frac{\alpha}{d}-\frac{|\gamma|}{d}},\ \gamma\in\mathbb{N}_0^d.$$

Theorem (X.-Xiong)

Let σ be a $\mathcal{M}$-valued symbol in class $S^0_{1,\delta}$, $0 \leq \delta < 1$ and $\alpha \geq 0$, then T^c_σ is a bounded operator from $F_1^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$ to itself.

Corollary

Let σ be a $\mathcal{M}$-valued symbol in class $S^m_{1,\delta}$, $0 \leq \delta < 1$ and $m \leq \alpha$, then T^c_σ is a bounded operator from $F_1^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$ to $F_1^{\alpha-m,c}(\mathbb{R}^d, \mathcal{M})$.

Thank you!