Characterization of compactness of commutators of bilinear Riesz transform

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Abstract: It is shown in this talk that the commutators of bilinear Riesz transforms and point-wise multiplication with a symbol in BMO are bilinear compact operators on product of Lebesgue spaces if and only if the symbol is actually in CMO. This extends the classical results of Coifman, Rochberg and Weiss and also of Uchiyama in linear setting to multi-linear setting.

- Joint work with

  ◇ Lucas Chaffee (Western Washington University, United States)

  ◇ Yanchang Han (South China Normal University, China)

  ◇ Rodolfo H. Torres (University of Kansas, United States)

  ◇ Lesley A. Ward (University of South Australia, Australia)

Linear setting

Commutator of Calderón-Zygmund operators

- Let T_Ω be the well-known Calderón-Zygmund singular integral operator defined by

$$T_\Omega f(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy,$$

where Ω satisfies
(i)$\Omega(\lambda x) = \Omega(x)$ for any $\lambda > 0$;
(ii) $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$;
(iii) $|\Omega(x') - \Omega(y')| \leq |x' - y'|$ for any $x', y' \in S^{n-1}$.

- Commutator of T_Ω with pointwise multiplication of function b is defined by

$$[b, T_\Omega]f(x) = b(x)T_\Omega f(x) - T_\Omega(bf)(x)$$

- Recall that the space of functions with bounded mean oscillation (BMO) consists of all locally integrable functions, b, such that

$$\|b\|_* := \sup_Q \int_Q |b(x) - b_Q| dx < \infty.$$

Theorem A (Coifman-Rochberg-Weiss, 1976)

The commutator $[b, T_\Omega]$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ if and only if $b \in BMO$.

- We define CMO space as the closure in BMO of the space of C^∞ functions with compact support.

Theorem B (Uchiyama, 1978)

The commutator $[b, T_\Omega]$ is a compact operator on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ if and only if $b \in CMO$.

Further developments

- From L^p spaces to weighted spaces, Morrey spaces, ...
- From classical Calderón-Zygmund operators to many important operators, such as, fractional integral operators, rough kernels, Bochner-Riesz operators, strongly singular integral operators, pseudo-differential operators, oscillatory kernel, Marcinkiewicz integral operators, ...
- From one parameter case to multi-parameter case
- From linear case to multi-linear case
- Shanzhen Lu, Silei Wang, Jiecheng Chen, Yong Ding, Guozhen Lu, Dachun Yang, Yanping Chen, Huoxiong Wu, Qingying Xue, ...
- Sagun Chanillo, Steve Hofmann, Michael T. Lacey, Jill Pipher, Rodolfo Torres, ...

Multilinear setting

Bilinear Calderón-Zygmund operators

- Let $K(x, y, z)$ be a locally integrable function defined away from the diagonal $x = y = z$. We say $K(x, y, z)$ is a bilinear Calderón-Zygmund kernel if for some parameters $A > 0$ and $\varepsilon > 0$ we have

$$|K(y_0, y_1, y_2)| \leq \frac{A}{(|y_0 - y_1| + |y_0 - y_2|)^{2n}}$$

and

$$|K(y_0, y_1, y_2) - K(y_0', y_1, y_2)| \leq \frac{A|y_0 - y_0'|^{\varepsilon}}{(|y_0 - y_1| + |y_0 - y_2|)^{2n+\varepsilon}}$$

whenever $|y_0 - y_0'| \leq \frac{1}{2}\max_{0\leq k\leq 2}|y_0 - y_k|$ and with similar estimates for y_1 and y_2. We say T is a bilinear Calderón-Zygmund operator if

$$T(f_1, f_2)(x) = \int\int K(x, y, z) f_1(y) f_2(z)\, dydz,$$

where K is a bilinear Calderón-Zygmund kernel.

Theorem C (Grafakos and Torres, 2002)

If T is a is a bilinear Calderón-Zygmund operator, then T is bounded from $L^{p_1} \times L^{p_2}$ to L^p with

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$$

for $1 \le p_1, p_2 < \infty$ and $1/2 < p < \infty$.

- Bilinear commutators of T are defined by

$$[b, T]_1(f, g)(x) := bT(f, g)(x) - T(bf, g)(x)$$

and

$$[b, T]_2(f, g)(x) := bT(f, g)(x) - T(f, bg)(x).$$

Theorem D (Pérez and Torres, 2003; Lerner et al., 2009)

If T is a is a bilinear Calderón-Zygmund operator and b belongs to BMO, then the commutators $[b, T]_1$ and $[b, T]_2$ are bounded from $L^{p_1} \times L^{p_2}$ to L^p with

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$$

for all $1 \leq p_1, p_2 < \infty$ and $1/2 < p < \infty$.

- Additional assumption on Calderón-Zygmund kernel K:
  (1) homogeneous of degree $-2n$, that is,

$$K(\lambda x, \lambda y, \lambda z) = \lambda^{-2n} K(x, y, z);$$

  (2) convolution type, that is, $K(x, y, z) = \widetilde{K}(x - y, x - z)$.

Theorem E (Chaffee, 2016)

If T is a is a bilinear Calderón-Zygmund operator and the kernel also satisfies the additional assumption (1) and (2) above. Assume that for local integrable function b, the commutators $[b, T]_1$ or $[b, T]_2$ is bounded from $L^{p_1} \times L^{p_2}$ to L^p with

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$$

for some $1 \leq p_1, p_2 < \infty$ and $1 < p < \infty$. Then b belongs to BMO.

Theorem F (Bényi and Torres, 2013; Ding-Mei-Xue, 2016)

If T is a is a bilinear Calderón-Zygmund operator and b belongs to CMO, then the commutators $[b, T]_1$ and $[b, T]_2$ are compact from $L^{p_1} \times L^{p_2}$ to L^p with

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$$

for all $1 < p_1, p_2 < \infty$ and $1 \leq p < \infty$.

Bilinear fractional integral operator

- For $0 < \alpha < 2n$, the bilinear fractional integral operator $\mathcal{I}_\alpha$ is a priori defined for $f, g \in C_c^\infty$ by

$$\mathcal{I}_\alpha(f,g)(x) := \int\int_{\mathbb{R}^{2n}} \frac{1}{(|x-y|+|x-z|)^{2n-\alpha}} f(y)g(z)dydz.$$

Theorem G (Chaffee and Torres, 2015)

Let $1 < p_1, p_2 < \infty$, $0 < \alpha < 2n$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n}$ with $1 < p < \infty$. Then the following are equivalent:
(1) $b \in CMO$;
(2) $[b, \mathcal{I}_\alpha]_1 : L^{p_1} \times L^{p_2} \to L^p$ is a compact operator.

Main results

Bilinear Riesz transform

- For $i = 1, 2$, let

$$K_i(x, y_1, y_2) := \frac{x - y_i}{(|x - y_1| + |x - y_2|)^{2n+1}}$$

Define bilinear Riesz transform R_i for $i = 1, 2$ by

$$R_i(f_1, f_2)(x) = \int\int K_i(x, y, z) f_1(y) f_2(z) dydz.$$

- For $i = 1, 2$, bilinear commutators of bilinear Riesz transform are defined by

$$[b, R_i]_1(f, g)(x) := bR_i(f, g)(x) - R_i(bf, g)(x)$$

and

$$[b, R_i]_2(f, g)(x) := bR_i(f, g)(x) - R_i(f, bg)(x).$$

Theorem 1 (Chaffee-Chen-Han-Torres-Ward, 2017)

Suppose $1 \leq i, \ell \leq 2$. *The following statement are equivalent:*

a) $b \in CMO(\mathbb{R}^n)$;

b) *the commutator* $[b, R_i]_\ell$ *is a compact operator mapping from* $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ *to* $L^p(\mathbb{R}^n)$ *for some (so for all)* $1 < p_1, p_2 < \infty$ *and* $1 \leq p < \infty$, *with*

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}.$$

Sketch of the proof (Uchiyama's approach)

- $b \in BMO$ is in CMO if and only if (Uchiyama, 1978)

$$\lim_{r\to 0} \sup_{|Q|=r^n} \frac{1}{|Q|}\int_Q |b(x)-b_Q|dx = 0;$$

$$\lim_{r\to \infty} \sup_{|Q|=r^n} \frac{1}{|Q|}\int_Q |b(x)-b_Q|dx = 0;$$

$$\lim_{R\to \infty} \sup_{Q\subset B(x_0,R)^c} \frac{1}{|Q|}\int_Q |b(x)-b_Q|dx = 0 \quad \text{for every fixed } x_0 \in \mathbb{R}^n.$$

Sketch of the proof

Lemma

Assume that $b \in BMO$ *and* $\|b\|_{BMO} = 1$*. Also assume that there exist* $\delta > 0$ *and a sequence of cubes* Q_j *such that for each* j*,*

$$\frac{1}{|Q_j|}\int_{Q_j} |b - b_{Q_j}|\, dx > \delta.$$

Then there exist sequences of functions $f_j \in L^{p_1}$ *and* $g_j \in L^{p_2}$ *with* $\|f_j\|_{L^{p_1}} \leq 1$ *and* $\|g_j\|_{L^{p_2}} \leq 1$*, positive constants* $A_1 > 4$*,* C_0, C_1 *and* C_2 *such that when* $k \geq [\log_2 A_1]$*, the following estimates hold:*

$$\int_{Q_j^k} \left|[b, R_1]_1(f_j, g_j)(y)\right|^p dy \geq C_1 \delta^p \frac{|Q_j|^{2p-1}}{|2^j Q_j|^{2p-1}};$$

$$\int_{2^{k+1}Q_j \setminus 2^k Q_j} \left|[b, R_1]_1(f_j, g_j)(y)\right|^p dy \leq C_2 \delta^p \frac{|Q_j|^{2p-1}}{|2^j Q_j|^{2p-1}};$$

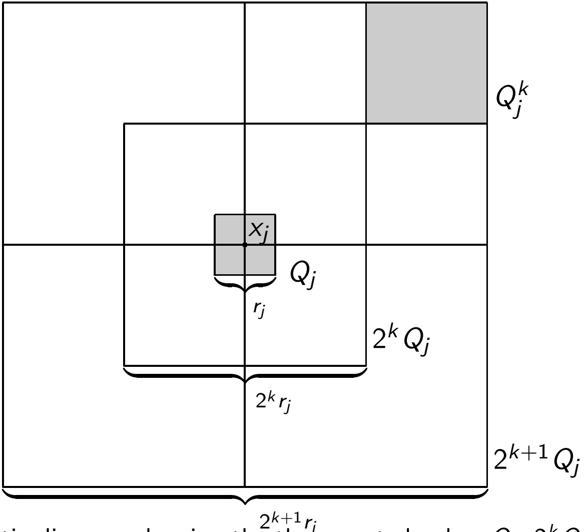

Figure: Schematic diagram showing the three nested cubes Q_j, $2^k Q_j$, and $2^{k+1} Q_j$ that are centered at x_j, and the cube Q_j^k in the top right corner of $2^{k+1} Q_j$. Not to scale.

Thank You !