Spaces and kernels

Marius Junge

UIUC

joint work with Minyu Zhao, and Sepideh Rezvani, and Qiang Zeng

Kernels

Kernels

❄ Integral operators are given by a **kernel**:

$$T_K(f)(x) = \int K(x,y)f(y)dy \, .$$

Kernels

- Integral operators are given by a **kernel**:

$$T_K(f)(x) = \int K(x,y)f(y)dy .$$

- Example 1: Fractional integrals $K(x,y) = |x-y|^{-(n+\alpha)}$.

Kernels

❄ Integral operators are given by a kernel:

$$T_K(f)(x) = \int K(x,y)f(y)dy\ .$$

❄ Example 1: Fractional integrals $K(x,y) = |x-y|^{-(n+\alpha)}$.

❄ Example 2: Singular integrals $|\nabla K(x,y)| \le |x-y|^{-\alpha}$

Kernels

- Integral operators are given by a kernel:

$$T_K(f)(x) \;=\; \int K(x,y)f(y)dy\;.$$

- Example 1: Fractional integrals $K(x,y) = |x-y|^{-(n+\alpha)}$.
- Example 2: Singular integrals $|\nabla K(x,y)| \;\leq\; |x-y|^{-\alpha}$
- The heat kernel of the semigroup $T_t = e^{-t|\Delta|}$ satisfies

$$K_t(x,y) \;=\; (2\pi t)^{-n/2} e^{-|x-y|^2/4t}\;.$$

* These estimates have been generalized to more general Riemanian manifolds using Ricci curvature and scalar curvature (see work of Coulhon, Saloff-Coste, and more).

- These estimates have been generalized to more general Riemanian manifolds using Ricci curvature and scalar curvature (see work of Coulhon, Saloff-Coste, and more).
- Varopoulos declares that a manifold (semigroup) has heat kernel dimension d if $\|T_t : L_1(M) \to L_\infty(M)\| \leq ct^{-d/2}$ and then proves an estimate for the resolvent

$$\|A^{-\alpha} : L_p(M) \to L_q(M)\|$$

- These estimates have been generalized to more general Riemanian manifolds using Ricci curvature and scalar curvature (see work of Coulhon, Saloff-Coste, and more).
- Varopoulos declares that a manifold (semigroup) has heat kernel dimension d if $\|T_t : L_1(M) \to L_\infty(M)\| \leq ct^{-d/2}$ and then proves an estimate for the resolvent

$$\|A^{-\alpha} : L_p(M) \to L_q(M)\|$$

- The Littlewood-Hardy operator is with kernel $|x-y|^{-(n+\alpha)}$ represents the resolvent $A^{-\alpha}$ of a semigroup $T_t = e^{-tA}$.

Kernels for von Neumann algebras

Kernels for von Neumann algebras

- Let N be a von Neumann algebra with normal semifinite trace tr, i.e. $tr : N_+ \to [0, \infty]$ is additive, $tr(u^* au) = tr(a)$ for unitaries and $\{a | tr(a^* a) < \infty\}$ is strongly dense.

Kernels for von Neumann algebras

- Let N be a von Neumann algebra with normal semifinite trace tr, i.e. $tr : N_+ \to [0, \infty]$ is additive, $tr(u^* au) = tr(a)$ for unitaries and $\{a | tr(a^* a) < \infty\}$ is strongly dense.
- For K affiliated with $N^{op} \bar{\otimes} N$, we define the integral operator

$$T_K(a) \; = \; tr \otimes id(K(a \otimes 1)) \, .$$

Wait a sec?

Wait a sec?

- Why the opposite algebra N^{op}?

Wait a sec?

- Why the opposite algebra N^{op}?
- What means heat kernel estimates? Can they be used to prove analytic results (for example for PDE), or the famous application to (local) index (theory)?

Wait a sec?

- Why the opposite algebra N^{op}?
- What means heat kernel estimates? Can they be used to prove analytic results (for example for PDE), or the famous application to (local) index (theory)?
- What is the connection to geometry and metric?

Wait a sec?

- Why the opposite algebra N^{op}?
- What means heat kernel estimates? Can they be used to prove analytic results (for example for PDE), or the famous application to (local) index (theory)?
- What is the connection to geometry and metric?
- Isoperimetric Inequalities?

Wait a sec?

- Why the opposite algebra N^{op}?
- What means heat kernel estimates? Can they be used to prove analytic results (for example for PDE), or the famous application to (local) index (theory)?
- What is the connection to geometry and metric?
- Isoperimetric Inequalities?
- What are singular integrals?

Wait a sec?

- Why the opposite algebra N^{op}?
- What means heat kernel estimates? Can they be used to prove analytic results (for example for PDE), or the famous application to (local) index (theory)?
- What is the connection to geometry and metric?
- Isoperimetric Inequalities?
- What are singular integrals? (see Javier's talk).

One page on Operator Spaces

One page on Operator Spaces

- Operator spaces are matrix normed subspaces of $B(H)$.

One page on Operator Spaces

- Operator spaces are matrix normed subspaces of $B(H)$. The category is closed under subspaces, quotients and duals.

One page on Operator Spaces

- Operator spaces are matrix normed subspaces of $B(H)$. The category is closed under subspaces, quotients and duals. The linear bounded maps are replaced by completely bounded maps with uniform control of matrix norms.

One page on Operator Spaces

- Operator spaces are matrix normed subspaces of $B(H)$. The category is closed under subspaces, quotients and duals. The linear bounded maps are replaced by completely bounded maps with uniform control of matrix norms.
- Effros-Ruan $CB(N_*, M) = N \bar{\otimes} M$.

One page on Operator Spaces

- Operator spaces are matrix normed subspaces of $B(H)$. The category is closed under subspaces, quotients and duals. The linear bounded maps are replaced by completely bounded maps with uniform control of matrix norms.
- Effros-Ruan $CB(N_*, M) = N\bar{\otimes}M$.
- Let N be a von Neumann algebra with a trace. With respect to $\langle x, y \rangle = tr(xy)$ the space $L_1(N) = N_*^{op}$ becomes the predual of the opposite algebra $a \cdot b = ba$-as operator space.

One page on Operator Spaces

- Operator spaces are matrix normed subspaces of $B(H)$. The category is closed under subspaces, quotients and duals. The linear bounded maps are replaced by completely bounded maps with uniform control of matrix norms.
- Effros-Ruan $CB(N_*, M) = N\bar{\otimes}M$.
- Let N be a von Neumann algebra with a trace. With respect to $\langle x, y\rangle = tr(xy)$ the space $L_1(N) = N_*^{op}$ becomes the predual of the opposite algebra $a \cdot b = ba$-as operator space.
- $CB(L_1(N), M) = N^{op}\bar{\otimes}M$.

One page on Operator Spaces

- Operator spaces are matrix normed subspaces of $B(H)$. The category is closed under subspaces, quotients and duals. The linear bounded maps are replaced by completely bounded maps with uniform control of matrix norms.
- Effros-Ruan $CB(N_*, M) = N\bar{\otimes}M$.
- Let N be a von Neumann algebra with a trace. With respect to $\langle x, y \rangle = tr(xy)$ the space $L_1(N) = N_*^{op}$ becomes the predual of the opposite algebra $a \cdot b = ba$-as operator space.
- $CB(L_1(N), M) = N^{op}\bar{\otimes}M$.
- Lesson learned: Heat kernel estimates can be considered in $N^{op}\bar{\otimes}N$ or in $N^{op} \otimes_\varepsilon N$.

Spaces

Spaces

- Let G be a discrete group and $L(G) = \lambda(G)''$ be the left regular representation $\lambda(g)e_h = e_{gh}$. Then we have an inclusion of the group algebra $\mathbb{C}[G] \subset C_{red}(G) = \overline{\lambda(\mathbb{C}(G))}^{\|\ \|}$ in the Reduced C^*-algebra.

Spaces

- Let G be a discrete group and $L(G) = \lambda(G)''$ be the left regular representation $\lambda(g)e_h = e_{gh}$. Then we have an inclusion of the group algebra $\mathbb{C}[G] \subset C_{red}(G) = \overline{\lambda(\mathbb{C}(G))}^{\| \ \|}$ in the Reduced C^*-algebra.
- Let θ_{jk} be an antisymmetric matrix. Then we may consider the rotation algebra A_θ^d generated by d unitaries u_k satisfying

$$u_k u_j \ = \ e^{i\theta_{jk}} u_j u_k \ .$$

Non-compact versions

Non-compact versions

- Let θ be an antisymmetric real matrix. We consider strongly continuous family of semigroups $u_k(t)$ such that

$$u_k(t)u_j(s) \;=\; e^{i\theta_{jk}st}u_j(s)u_k(t)\;.$$

Non-compact versions

- Let θ be an antisymmetric real matrix. We consider strongly continuous family of semigroups $u_k(t)$ such that

$$u_k(t)u_j(s) = e^{i\theta_{jk}st}u_j(s)u_k(t) .$$

Let $\lambda_\theta(\xi) = u_1(\xi_1)\cdots u_d(\xi_d)$ and for a Schwarz function f we define

$$\lambda_\theta(f) = \int \lambda_\theta(\xi)f(\xi)d\xi .$$

Non-compact versions

- Let θ be an antisymmetric real matrix. We consider strongly continuous family of semigroups $u_k(t)$ such that

$$u_k(t)u_j(s) = e^{i\theta_{jk}st}u_j(s)u_k(t) .$$

Let $\lambda_\theta(\xi) = u_1(\xi_1)\cdots u_d(\xi_d)$ and for a Schwarz function f we define

$$\lambda_\theta(f) = \int \lambda_\theta(\xi)f(\xi)d\xi .$$

- We denote by $E_\theta \cong C(\mathbb{R}^d_\theta)$ the C^*-algebra of continuous functions.

Non-compact versions

- Let θ be an antisymmetric real matrix. We consider strongly continuous family of semigroups $u_k(t)$ such that

$$u_k(t)u_j(s) = e^{i\theta_{jk}st}u_j(s)u_k(t) .$$

Let $\lambda_\theta(\xi) = u_1(\xi_1)\cdots u_d(\xi_d)$ and for a Schwarz function f we define

$$\lambda_\theta(f) = \int \lambda_\theta(\xi)f(\xi)d\xi .$$

- We denote by $E_\theta \cong C(\mathbb{R}^d_\theta)$ the C^*-algebra of continuous functions. This is the C^*-version of our quantum euclidean space.

Traces-and crossed products

Traces-and crossed products

- R^d_θ is the von Neumann algebra version of the quantum euclidean space given by the GNS construction with respect to

$$tr_\theta(\lambda_\theta(f)) = f(0) .$$

Traces-and crossed products

- R_θ^d is the von Neumann algebra version of the quantum euclidean space given by the GNS construction with respect to

$$tr_\theta(\lambda_\theta(f)) = f(0) \, .$$

- $R_\theta = L_\infty(\mathbb{R}) \rtimes_{\alpha_1} \rtimes \cdots \rtimes \mathbb{R}$ can be identified as an iterated crossed product.

Traces-and crossed products

- R^d_θ is the von Neumann algebra version of the quantum euclidean space given by the GNS construction with respect to

$$tr_\theta(\lambda_\theta(f)) \;=\; f(0)\,.$$

- $R_\theta = L_\infty(\mathbb{R}) \rtimes_{\alpha_1} \rtimes \cdots \rtimes \mathbb{R}$ can be identified as an iterated crossed product.
- For a $\mathbb{Z}^2$ action on a C^*-algebra B we define a new crossed product

$$B \rtimes A_\theta$$

as the C^* algebra generated by $\pi(B)$ given by $\pi(b) = \sum_{k\in\mathbb{Z}} e_{kk} \otimes \alpha_k^{-1}(b)$ and

$$U_\theta(k_1,k_2) = \lambda((k_1,k_2))\otimes u_1(k_1)u_2(k_2) \in B(H\otimes\ell_2(\mathbb{Z})\otimes L_2(A^2_\theta)\,.$$

Finite dimensional version

Finite dimensional version

- Let $d = 2$, $n \in \mathbb{N}$. Then we can consider the diagonal matrix $u(e_r) = e^{\frac{2\pi ri}{n}} e_r$ and the shift matrix $v(e_r) = e_{r+1}$. They satisfy $u^n = 1 = v^n$ and

$$uv = e^{\frac{2\pi i}{n}} vu .$$

Finite dimensional version

- Let $d = 2$, $n \in \mathbb{N}$. Then we can consider the diagonal matrix $u(e_r) = e^{\frac{2\pi ri}{n}} e_r$ and the shift matrix $v(e_r) = e_{r+1}$. They satisfy $u^n = 1 = v^n$ and

$$uv = e^{\frac{2\pi i}{n}} vu .$$

- Similarly, we can realize $u_1^{k_1} u_2^{k_2} = e^{i\theta k_1 k_2} u_2^{k_2} u_1^{k_1}$ in the space of $n \times n$ matrices if $\theta = \frac{2\pi l}{n}$. Let us call this algebra $A_\theta(n)$.

Finite dimensional version

- Let $d = 2$, $n \in \mathbb{N}$. Then we can consider the diagonal matrix $u(e_r) = e^{\frac{2\pi ri}{n}} e_r$ and the shift matrix $v(e_r) = e_{r+1}$. They satisfy $u^n = 1 = v^n$ and

$$uv = e^{\frac{2\pi i}{n}} vu .$$

- Similarly, we can realize $u_1^{k_1} u_2^{k_2} = e^{i\theta k_1 k_2} u_2^{k_2} u_1^{k_1}$ in the space of $n \times n$ matrices if $\theta = \frac{2\pi l}{n}$. Let us call this algebra $A_\theta(n)$.
- For $\mathbb{Z}_n^2$ actions we may define the discrete crossed product $B \rtimes A_\theta(n)$ as above.

Finite dimensional version

- Let $d = 2$, $n \in \mathbb{N}$. Then we can consider the diagonal matrix $u(e_r) = e^{\frac{2\pi ri}{n}} e_r$ and the shift matrix $v(e_r) = e_{r+1}$. They satisfy $u^n = 1 = v^n$ and

$$uv = e^{\frac{2\pi i}{n}} vu .$$

- Similarly, we can realize $u_1^{k_1} u_2^{k_2} = e^{i\theta k_1 k_2} u_2^{k_2} u_1^{k_1}$ in the space of $n \times n$ matrices if $\theta = \frac{2\pi l}{n}$. Let us call this algebra $A_\theta(n)$.
- For $\mathbb{Z}_n^2$ actions we may define the discrete crossed product $B \rtimes A_\theta(n)$ as above.
- (Technical tool) Finite dimensional rotation algebras $A_\theta^{2d}(n)$ can be viewed as iterated cross products

$$A_\theta^{2d}(n) = A_{\theta(1)}(n) \rtimes \cdots \rtimes A_{\theta(d)}(n) .$$

Laplacians

Laplacians

- On A_θ^d the semigroup $T_t(u_1(k_1)\cdots u_d(k_d)) = e^{-t(k_1^2+\cdots k_d^2)}$ has the generator $\Delta(u_1(k_1)\cdots u_d(k_d)) = |(k_1, ..., k_d)|^2 u_1(k_1)\cdots u_d(k_d)$.

Laplacians

- On A_θ^d the semigroup $T_t(u_1(k_1)\cdots u_d(k_d)) = e^{-t(k_1^2+\cdots k_d^2)}$ has the generator $\Delta(u_1(k_1)\cdots u_d(k_d)) = |(k_1,...,k_d)|^2 u_1(k_1)\cdots u_d(k_d)$.
- On R_θ^d the semigroup $T_t(\lambda_\theta(\xi)) = e^{-t\|\xi\|^2}\lambda_\theta(\xi)$ with generator $\Delta(\lambda_\theta(\xi)) = \|\xi\|^2\lambda_\theta(\xi)$.

Laplacians

- On A_θ^d the semigroup $T_t(u_1(k_1)\cdots u_d(k_d)) = e^{-t(k_1^2+\cdots k_d^2)}$ has the generator $\Delta(u_1(k_1)\cdots u_d(k_d)) = |(k_1, \ldots, k_d)|^2 u_1(k_1)\cdots u_d(k_d)$.
- On R_θ^d the semigroup $T_t(\lambda_\theta(\xi)) = e^{-t\|\xi\|^2}\lambda_\theta(\xi)$ with generator $\Delta(\lambda_\theta(\xi)) = \|\xi\|^2\lambda_\theta(\xi)$.
- For a discrete group a semigroup of Fourier-multipliers is given by $T_t(\lambda(g)) = e^{-t\psi(g)}$ so that $\psi(g)$ is conditionally positive.

Laplacians

- On A_θ^d the semigroup $T_t(u_1(k_1)\cdots u_d(k_d)) = e^{-t(k_1^2+\cdots k_d^2)}$ has the generator $\Delta(u_1(k_1)\cdots u_d(k_d)) = |(k_1, ..., k_d)|^2 u_1(k_1)\cdots u_d(k_d)$.
- On R_θ^d the semigroup $T_t(\lambda_\theta(\xi)) = e^{-t\|\xi\|^2}\lambda_\theta(\xi)$ with generator $\Delta(\lambda_\theta(\xi)) = \|\xi\|^2\lambda_\theta(\xi)$.
- For a discrete group a semigroup of Fourier-multipliers is given by $T_t(\lambda(g)) = e^{-t\psi(g)}$ so that $\psi(g)$ is conditionally positive.
- For $\mathbb{Z}_n$ we use $\psi(k) = \frac{n}{\pi}(1 - \cos(\frac{2\pi k}{n}))$-

Laplacians

- On A_θ^d the semigroup $T_t(u_1(k_1)\cdots u_d(k_d)) = e^{-t(k_1^2+\cdots k_d^2)}$ has the generator $\Delta(u_1(k_1)\cdots u_d(k_d)) = |(k_1,...,k_d)|^2 u_1(k_1)\cdots u_d(k_d)$.
- On R_θ^d the semigroup $T_t(\lambda_\theta(\xi)) = e^{-t\|\xi\|^2}\lambda_\theta(\xi)$ with generator $\Delta(\lambda_\theta(\xi)) = \|\xi\|^2\lambda_\theta(\xi)$.
- For a discrete group a semigroup of Fourier-multipliers is given by $T_t(\lambda(g)) = e^{-t\psi(g)}$ so that $\psi(g)$ is conditionally positive.
- For $\mathbb{Z}_n$ we use $\psi(k) = \frac{n}{\pi}(1-\cos(\frac{2\pi k}{n}))$-gives correct limit at $n=\infty$.

Laplacians

- On A_θ^d the semigroup $T_t(u_1(k_1)\cdots u_d(k_d)) = e^{-t(k_1^2+\cdots k_d^2)}$ has the generator $\Delta(u_1(k_1)\cdots u_d(k_d)) = |(k_1,...,k_d)|^2 u_1(k_1)\cdots u_d(k_d)$.
- On R_θ^d the semigroup $T_t(\lambda_\theta(\xi)) = e^{-t\|\xi\|^2}\lambda_\theta(\xi)$ with generator $\Delta(\lambda_\theta(\xi)) = \|\xi\|^2\lambda_\theta(\xi)$.
- For a discrete group a semigroup of Fourier-multipliers is given by $T_t(\lambda(g)) = e^{-t\psi(g)}$ so that $\psi(g)$ is conditionally positive.
- For $\mathbb{Z}_n$ we use $\psi(k) = \frac{n}{\pi}(1-\cos(\frac{2\pi k}{n}))$-gives correct limit at $n=\infty$.
- Similarly for $A_\theta(n)$ using co-multiplication or $\mathbb{Z}_n^2$ action.

Diagonals-simple observation

Diagonals-simple observation

* The map $\pi : L_\infty(\mathbb{R}^d) \to R_\theta \bar{\otimes} R_\theta^{op}$ given by

$$\pi(\lambda_0(\xi)) \;=\; \lambda_\theta(\xi) \otimes \lambda_\theta(\xi)^*$$

extends to a normal *-homomorphism.

Diagonals-simple observation

* The map $\pi : L_\infty(\mathbb{R}^d) \to R_\theta \bar{\otimes} R_\theta^{op}$ given by

$$\pi(\lambda_0(\xi)) = \lambda_\theta(\xi) \otimes \lambda_\theta(\xi)^*$$

extends to a normal *-homomorphism.

* Similarly, $\pi : L_\infty(\mathbb{T}^d) \to A_\theta \otimes A_\theta^{op}$ given by

$$\pi(\lambda_0(k)) = u_\theta(k) \otimes u_\theta(k)^*$$

extends to a normal *-homomorphism.

Kernels $\Rightarrow$ Operators

Kernels $\Rightarrow$ Operators

- $K_1, K_2 \in N \otimes N^{op}$, then the composition satisfies

$$T_{K_1} \circ T_{K_2} = T_{K_1 \cdot K_2}$$

Kernels ⇒ Operators

- $K_1, K_2 \in N \otimes N^{op}$, then the composition satisfies

$$T_{K_1} \circ T_{K_2} \ = \ T_{K_1 \cdot K_2}$$

- For k affiliated to $L_\infty(\mathbb{R}^d)$, we see that the operator $T_{\pi(k)}$ is given by the Fourier multiplier $F_{\hat{k}}(\lambda_\theta(\xi)) = \hat{k}(\xi)\lambda_\theta(\xi)$.

Applications

Theorem The heat kernel estimate

$$\|T_t : L_1(N) \to L_\infty(N)\|_{cb} \le c(d) t^{-d/2}$$

holds for $N = R^d_\theta$, $N = A^d_\theta$, and $A^d_\theta(n)$.

Proof: Let $k_t(x) = (4\pi d t)^{-d/2} e^{-|x|^2/4t}$. Then

$$\|T_t\|_{cb} = \|\pi(k_t)\|_{N^{op}\bar{\otimes}N} \le \|k_t\|_\infty .$$

Applications

Theorem The heat kernel estimate

$$\|T_t : L_1(N) \to L_\infty(N)\|_{cb} \ \le \ c(d)t^{-d/2}$$

holds for $N = R^d_\theta$, $N = A^d_\theta$, and $A^d_\theta(n)$.

Proof: Let $k_t(x) = (4\pi dt)^{-d/2} e^{-|x|^2/4t}$. Then

$$\|T_t\|_{cb} \ = \ \|\pi(k_t)\|_{N^{op}\bar{\otimes} N} \ \le \ \|k_t\|_\infty \ .$$

Remark: The corresponding Strichartz estimates $\|e^{it\Delta} : L_1 \to L_\infty\| \le ct^{-d/2}$ holds, and hence Tao-Keel's work implies well-posedness for short time solutions. for Quantum Euclidean Spaces.

Metric aspects

Metric aspects

- Following Connes, Rieffel studies quantum metric spaces.

Metric aspects

- Following Connes, Rieffel studies quantum metric spaces. This means he considers triples $(\mathcal{A}, A, \| \|_{Lip})$ where $\mathcal{A} \subset A$ is a * algebra of a C^*-algebra A and the inclusion map $id : (\mathcal{A}, \| \|_{Lip}) \to (A, \| \|_A)$ is compact.

Metric aspects

- Following Connes, Rieffel studies quantum metric spaces. This means he considers triples $(\mathcal{A}, A, \| \|_{Lip})$ where $\mathcal{A} \subset A$ is a * algebra of a C^*-algebra A and the inclusion map $id : (\mathcal{A}, \| \|_{Lip}) \to (A, \| \|_A)$ is compact. This implies that

$$d_{Lip}(\varphi, \psi) = \sup_{\|a\|_{Lip} \leq 1} |\varphi(a) - \psi(a)|$$

induces the weak* topology on the state space.

Metric aspects

- Following Connes, Rieffel studies quantum metric spaces. This means he considers triples $(\mathcal{A}, A, \| \|_{Lip})$ where $\mathcal{A} \subset A$ is a * algebra of a C^*-algebra A and the inclusion map $id : (\mathcal{A}, \| \|_{Lip}) \to (A, \| \|_A)$ is compact. This implies that

$$d_{Lip}(\varphi, \psi) = \sup_{\|a\|_{Lip} \le 1} |\varphi(a) - \psi(a)|$$

induces the weak* topology on the state space.

- Rieffel and Latremoliere study different definitions in order to verify statements from the Physics literature showing that sphere's are Gromov-Hausdorff limits of matrix algebras.

Previous work

Previous work

- Rieffel (Li) show that a sphere can be approximated by matrix algebras (confirming claims by physicists) using the notion of Gromov-Hausdorff space for order unit spaces.

Previous work

- Rieffel (Li) show that a sphere can be approximated by matrix algebras (confirming claims by physicists) using the notion of Gromov-Hausdorff space for order unit spaces.
- Latremoliere (following Rieffel, Li) shows that with most of his definitions this is also true for Rotation algebras and an approximation by Matrix algebras with 'their' definition of quantum metric-'propinquity' (this has only two meanings with and without dual).

Semigroup-Lip norms

Semigroup-Lip norms

- Let A be a generator of a semigroup and consider the gradient form (carrë du champs)

$$2\Gamma(x,y) \ = \ A(x^*)y + x^*A(y) - A(x^*y) \ .$$

Semigroup-Lip norms

- Let A be a generator of a semigroup and consider the gradient form (carrë du champs)

$$2\Gamma(x,y) \;=\; A(x^*)y + x^*A(y) - A(x^*y)\,.$$

- For $A = -\Delta$, we find $\Gamma(f,g) = (\nabla f, \nabla g)$.

Semigroup-Lip norms

- Let A be a generator of a semigroup and consider the gradient form (carrë du champs)

$$2\Gamma(x,y) \;=\; A(x^*)y + x^*A(y) - A(x^*y)\;.$$

- For $A=-\Delta$, we find $\Gamma(f,g)=(\nabla f,\nabla g)$.
- There exists a derivation δ on $\mathrm{dom}(A^{1/2})$ with values in a Hilbert C^*-bimodule such that

$$(\delta(x),\delta(y)) \;=\; \Gamma(x,y)\;.$$

Semigroup-Lip norms

- Let A be a generator of a semigroup and consider the gradient form (carrë du champs)

$$2\Gamma(x,y) \;=\; A(x^*)y + x^*A(y) - A(x^*y)\,.$$

- For $A = -\Delta$, we find $\Gamma(f,g) = (\nabla f, \nabla g)$.
- There exists a derivation δ on $\mathrm{dom}(A^{1/2})$ with values in a Hilbert C^*-bimodule such that

$$(\delta(x), \delta(y)) \;=\; \Gamma(x,y)\,.$$

- According to [JRS], δ can be viewed with values in $L_2(M)$ and inner product given by a conditional expectation for some vNa M.

Semigroup-Lip norms

- Let A be a generator of a semigroup and consider the gradient form (carrë du champs)

$$2\Gamma(x,y) \;=\; A(x^*)y + x^*A(y) - A(x^*y)\,.$$

- For $A = -\Delta$, we find $\Gamma(f,g) = (\nabla f, \nabla g)$.
- There exists a derivation δ on $\mathrm{dom}(A^{1/2})$ with values in a Hilbert C^*-bimodule such that

$$(\delta(x), \delta(y)) \;=\; \Gamma(x,y)\,.$$

- According to [JRS], δ can be viewed with values in $L_2(M)$ and inner product given by a conditional expectation for some vNa M.
- $\|\nabla x\|_p = \max\{\|\Gamma(x,x)^{1/2}\|_p, \|\Gamma(x^*,x^*)^{1/2}\|_p\}$ satisfies

$$\|\nabla(ax)\|_p \;\le\; \|\nabla(a)\|_p\|x\|_\infty + \|a\|_\infty\|\nabla(x)\|_p\,.$$

CB-version of Gromov-Hausdorff convergence

Theorem Let us define an operator space 'via' $\|x\|_{Lip} = \|\nabla(x)\|_\infty$. There exist a sequence $B_n = A^{2d}_{\theta_n}(k_n)$ of matrix algebras, and *-homomorphisms $\pi_n : B_n \to \mathbb{B}(H)$, $\pi_\infty : A^{2d}_\Theta \to \mathbb{B}(H)$ with the following convergence property.

CB-version of Gromov-Hausdorff convergence

Theorem Let us define an operator space 'via' $\|x\|_{Lip} = \|\nabla(x)\|_\infty$. There exist a sequence $B_n = A^{2d}_{\theta_n}(k_n)$ of matrix algebras, and $*$-homomorphisms $\pi_n : B_n \to \mathbb{B}(H)$, $\pi_\infty : A^{2d}_\Theta \to \mathbb{B}(H)$ with the following convergence property. For every $\varepsilon > 0$ there exists an n_ε such that for all $n > n_\varepsilon$

I) for every $x \in \mathcal{K} \otimes A^{2d}_\Theta$ with $\|x\|_{Lip} \leq 1$, there exists a $y \in \mathcal{K} \otimes B_n$ with $\|y\|_{Lip} \leq 1$ and

$$\|\pi_\infty(x) - \pi_n(y)\| \leq \varepsilon \, ,$$

II) for every $x \in \mathcal{K} \otimes B_n$ with $\|x\|_{Lip} \leq 1$, there exists a $y \in \mathcal{K} \otimes A^{2d}_\Theta$ with $\|y\|_{Lip} \leq 1$ and

$$\|\pi_n(x) - \pi_\infty(y)\| \leq \varepsilon \, .$$

The proof uses ideas of Li, Kerr Latremoliere, Kirchberg, Blanchard, and Junge-Mei (uniform harmonic analysis in matrix algebras).

The proof uses ideas of Li, Kerr Latremoliere, Kirchberg, Blanchard, and Junge-Mei (uniform harmonic analysis in matrix algebras).

$$\begin{array}{ccc} B_n & & B_\infty = A_\theta \\ & \searrow \quad \swarrow & \\ & B(H) & \end{array}$$

so that $\pi_n(\mathrm{Ball}(\mathcal{K} \otimes B_n), \| \, \|_{Lip})$ and $\pi_\infty(\mathrm{Ball}(\mathcal{K} \otimes A_\theta), \| \, \|_{Lip})$ are close in the Hausdorff distance.

Connes' derivations

Connes' derivations

- For $d = 2$ and A_θ spanned by $uv = e^{i\theta} vu$, Connes defines derivations

$$\delta_1(u) = 1 = \delta_2(v) \, , \; \delta_1(v) = 0 = \delta_2(v) \, .$$

Then the Dirac operator is given by $D = \delta_1 + i\delta_2$.

Connes' derivations

- For $d = 2$ and A_θ spanned by $uv = e^{i\theta} vu$, Connes defines derivations

$$\delta_1(u) = 1 = \delta_2(v) \, , \; \delta_1(v) = 0 = \delta_2(v) \, .$$

Then the Dirac operator is given by $D = \delta_1 + i\delta_2$.

- Then $\|[D, x]\|_p \sim \|\nabla(x)\|_p$.

Connes' derivations

- For $d = 2$ and A_θ spanned by $uv = e^{i\theta} vu$, Connes defines derivations

$$\delta_1(u) = 1 = \delta_2(v)\ ,\ \delta_1(v) = 0 = \delta_2(v)\ .$$

Then the Dirac operator is given by $D = \delta_1 + i\delta_2$.

- Then $\|[D, x]\|_p \sim \|\nabla(x)\|_p$.
- This remains true for R^θ and $D = \sum_j c_j \delta_j$, where $\delta_j(\lambda_\theta(\xi)) = i\xi_j \lambda_\theta(\xi)$ is the canonical inner derivation. (

Connes' derivations

- For $d = 2$ and A_θ spanned by $uv = e^{i\theta}vu$, Connes defines derivations

$$\delta_1(u) = 1 = \delta_2(v)\ ,\ \delta_1(v) = 0 = \delta_2(v)\ .$$

  Then the Dirac operator is given by $D = \delta_1 + i\delta_2$.
- Then $\|[D,x]\|_p \sim \|\nabla(x)\|_p$.
- This remains true for R^θ and $D = \sum_j c_j\delta_j$, where $\delta_j(\lambda_\theta(\xi)) = i\xi_j\lambda_\theta(\xi)$ is the canonical inner derivation. (For general von Neumann algebras these derivations should have values in bimodules).

Riesz transforms and Mihlin-multiplier

Riesz transforms and Mihlin-multiplier

- Adapting results from [JM,JPM], we find that

$$\| \sum_{j=1}^{d} g_j \delta_j(f) \|_p \sim_{c(p)} \|A^{1/2} x\|_p \, .$$

Riesz transforms and Mihlin-multiplier

- Adapting results from [JM,JPM], we find that

$$\|\sum_{j=1}^{d} g_j \delta_j(f)\|_p \sim_{c(p)} \|A^{1/2}x\|_p \,.$$

- This also holds for fractional powers $A^{\alpha/2}$ of the Laplacian, but the statement for $p < 2$ has to be modified.

Riesz transforms and Mihlin-multiplier

- Adapting results from [JM,JPM], we find that

$$\| \sum_{j=1}^{d} g_j \delta_j(f) \|_p \sim_{c(p)} \|A^{1/2} x\|_p \,.$$

- This also holds for fractional powers $A^{\alpha/2}$ of the Laplacian, but the statement for $p < 2$ has to be modified.
- (follows from [JMP]) Let $m : \mathbb{R}^n \to \mathbb{C}$ be $\lfloor \frac{d+1}{2} \rfloor$-times differentiable with bounded derivatives. Then the Fourier multiplier $F_m(\lambda_\theta(\xi)) = m(\xi)\lambda_\theta(\xi)$ extends to a (completely) bounded operator on $L_p(\mathbb{R}^d_\theta)$, $1 < p < \infty$.

Resolvent estimates for Orlicz-norms

Resolvent estimates for Orlicz-norms

- We follow a well-known road map: Varopoulos, Saloff-Coste, Boudoin,...

Resolvent estimates for Orlicz-norms

- We follow a well-known road map: Varopoulos, Saloff-Coste, Boudoin,...
- For a convex function Φ with $\Phi(0) = 0$, we define

$$\|x\|_{\Phi} = \inf\{\lambda | tr(\Phi(\frac{|x|}{\lambda})) \leq 1\}$$

Resolvent estimates for Orlicz-norms

- We follow a well-known road map: Varopoulos, Saloff-Coste, Boudoin,...
- For a convex function Φ with $\Phi(0) = 0$, we define

$$\|x\|_\Phi = \inf\{\lambda | tr(\Phi(\frac{|x|}{\lambda})) \leq 1\}$$

- We assume (g) that

$$\|T_t : L_1(N) \to L_\infty(N)\| \leq g(t) .$$

Resolvent estimates for Orlicz-norms

- We follow a well-known road map: Varopoulos, Saloff-Coste, Boudoin,...
- For a convex function Φ with $\Phi(0) = 0$, we define

$$\|x\|_\Phi \;=\; \inf\{\lambda | tr(\Phi(\frac{|x|}{\lambda})) \le 1\}$$

- We assume (g) that

$$\|T_t : L_1(N) \to L_\infty(N)\| \;\le\; g(t)\,.$$

- Recall that for $T_t = e^{-tA}$, we have

$$A^{-\alpha}(x) \;=\; \Gamma(\alpha)^{-1} \int_0^\infty t^{\alpha-1} T_t(x)dt\,.$$

Resolvent estimates for Orlicz-norms

- We follow a well-known road map: Varopoulos, Saloff-Coste, Boudoin,...
- For a convex function Φ with $\Phi(0) = 0$, we define

$$\|x\|_\Phi \; = \; \inf\{\lambda | tr(\Phi(\frac{|x|}{\lambda})) \leq 1\}$$

- We assume (g) that

$$\|T_t : L_1(N) \to L_\infty(N)\| \; \leq \; g(t) \, .$$

- Recall that for $T_t = e^{-tA}$, we have

$$A^{-\alpha}(x) \; = \; \Gamma(\alpha)^{-1} \int_0^\infty t^{\alpha-1} T_t(x) dt \, .$$

- Varopoulos (Badauin) showed that heat kernel estimates imply resolvent estimates.

Idea

Idea

- Split integral

$$\int_0^\infty t^{\alpha-1} T_t(x)dt = \int_0^{b(\omega)} t^{\alpha-1} T_t(x)dt + \int_{b(\omega)}^\infty t^{\alpha-1} T_t(x)dt$$

as a random function.

Idea

- Split integral

$$\int_0^\infty t^{\alpha-1} T_t(x)dt = \int_0^{b(\omega)} t^{\alpha-1} T_t(x)dt + \int_{b(\omega)}^\infty t^{\alpha-1} T_t(x)dt$$

  as a random function.
- Noncommutative idea: Use the the upper bound $T_t(x) \leq a$ from Doob's inequality. Assume $a = \sum_k a_k e_k$ with $\sum_k e_k = 1$.

Idea

- Split integral

$$\int_0^\infty t^{\alpha-1} T_t(x)dt \;=\; \int_0^{b(\omega)} t^{\alpha-1} T_t(x)dt + \int_{b(\omega)}^\infty t^{\alpha-1} T_t(x)dt$$

as a random function.

- Noncommutative idea: Use the the upper bound $T_t(x) \le a$ from Doob's inequality. Assume $a = \sum_k a_k e_k$ with $\sum_k e_k = 1$. Choose $b(k)$. Then the function $F_{kj}(t) = e_k T_t(x) e_j t^{\alpha-1}$ admits a decomposition

$$\begin{aligned} F_{kj}(t) &= 1_{[0,b_k]}(t) F_{kj}(t) 1_{[0,b_j]}(t) + 1_{[b_k,\infty]}(t) F_{kj}(t) 1_{[b_j,\infty]}(t) \\ &\quad + 1_{[0,b_k]}(t) F_{kj}(t) 1_{[b_j,\infty]}(t) + 1_{[b_k,\infty]}(t) F_{kj}(t) 1_{[0,b_j]}(t) \end{aligned}$$

Result

Theorem (J.-Zhao) Let $g, h : [0, \infty) \to [0, \infty)$ be increasing, respectively decreasing so that $\psi(x) = xh(x)g(x)^{1-p}$ is ing. Let $\varphi_{g,h}(w) = wh^{-1}(w)gh^{-1}(w)$ and $\Phi(x) = \varphi_{g,h}^{-1}(x)^{1/p}$.

Result

Theorem (J.-Zhao) Let $g, h : [0, \infty) \to [0, \infty)$ be increasing, respectively decreasing so that $\psi(x) = xh(x)g(x)^{1-p}$ is inc. Let $\varphi_{g,h}(w) = wh^{-1}(w)gh^{-1}(w)$ and $\Phi(x) = \varphi_{g,h}^{-1}(x)^{1/p}$. Then the Laplace transform

$$\mathcal{L}_g(A) = \int_0^\infty T_t g(t)dt$$

is bounded from

$$\|\mathcal{L}_g(A) : L_p \to L_\Phi\|$$

whenever T_t is a semigroup of completely positive selfadjoint maps on a von Neumann algebra.

Typical Example:

We have

$$\| T_t : L_p \to L_\infty \| \leq \begin{cases} t^{-d_1/2p} & 0 < t \leq 1 \\ t^{-d_2/2p} & 1 < t < \infty \end{cases}$$

Then

$$A^{-\alpha} : L_p \to L_{q_1} + L_{q_2} \ .$$

Typical Example:

We have

$$\|T_t : L_p \to L_\infty\| \le \begin{cases} t^{-d_1/2p} & 0 < t \le 1 \\ t^{-d_2/2p} & 1 < t < \infty \end{cases}$$

Then

$$A^{-\alpha} : L_p \to L_{q_1} + L_{q_2} \ .$$

Similarly, for $g(t) = \max(\min)\{t^{\alpha_1 - 1}, t^{\alpha_2 - 1}\}$, we obtain estimates with values in $L_{q_1} + L_{q_2}$ or $L_{q_1} \cap L_{q_2}$ (calculus).

Open Problems

Open Problems

- Show that the fractional integral

$$A^{-\alpha}(f) = c_\alpha \int \frac{f(y)}{|x-y|^{n+\alpha}} dy$$

is completely bounded between L_p and $L_{q(\alpha,d)}$.

Open Problems

- Show that the fractional integral

$$A^{-\alpha}(f) = c_\alpha \int \frac{f(y)}{|x-y|^{n+\alpha}} dy$$

is completely bounded between L_p and $L_{q(\alpha,d)}$.

- Is it true that for Fourier multipliers $m : \mathbb{R}^d \to \mathbb{C}$, we have

$$\|F_m : L_p(\mathbb{R}^d_\theta) \to L_p(\mathbb{R}^d_\theta)\|_{cb} = \|F_m L_p(\mathbb{R}^d) \to L_p(\mathbb{R}^d)\|_{cb}$$

Holds for A_θ!

Open Problems

- Show that the fractional integral

$$A^{-\alpha}(f) = c_\alpha \int \frac{f(y)}{|x-y|^{n+\alpha}} dy$$

  is completely bounded between L_p and $L_{q(\alpha,d)}$.
- Is it true that for Fourier multipliers $m : \mathbb{R}^d \to \mathbb{C}$, we have

$$\|F_m : L_p(\mathbb{R}^d_\theta) \to L_p(\mathbb{R}^d_\theta)\|_{cb} = \|F_m L_p(\mathbb{R}^d) \to L_p(\mathbb{R}^d)\|_{cb}$$

  Holds for A_θ!
- Study Riesz transform estimates for conformal change of metrics (see Connes-Tretkoff, states).

Thanks Guixiang and Tao

Thanks Guixiang and Tao

Thanks for listening