# High Order Eigenvalues for Non-Local Schrödinger Operators

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# Outline



- ♣ Functional Inequalities for Discrete Spectrum
- ♣ Non-Local Schrödinger Operators
- ♣ Lower Bound Estimates on Eigenvalues
- ♣ Upper Bound Estimates on Eigenvalues
- ♣ Example

# Functional Inequalities for Discrete Spectrum

Framework:

- $(E, \mathscr{F}, \mu)$: a $\sigma$-finite separable measurable space.
- $(L, \mathscr{D}(L))$: self-adjoint operator on $L^2(\mu)$ generating a $C_0$-contraction semigroup $P_t := \mathrm{e}^{tL}$.
- Energy form $\mathscr{E}(f,f) := -\int_E fLf\mathrm{d}\mu \geq 0,\ f \in \mathscr{D}(L)$, extends to $f \in \mathscr{D}(\mathscr{E}) = \mathscr{D}(\sqrt{-L})$.
- $\sigma_{ess}(L)$: the essential spectrum of $L$.
- When $\sigma_{ess}(L) = \emptyset$, let $0 \leq \lambda_1 \leq \lambda_2 \cdots \leq \lambda_n \cdots \uparrow \infty$ be all eigenvalues of $-L$.

We assume that $P_t$ has a heat kernel $p_t(x,y)$ for some $t > 0$:

$$P_t f(x) = \int_E p_t(x,y) f(y) \mu(\mathrm{d}y), \quad f \in L^2(\mu).$$

This condition can be slightly weakened as

Exietnce of asymptotic heat kernel

# Functional Inequalities for Discrete Spectrum

**Theorem (W02'JFA)**

*Let $r_0 \geq 0$. Then $\sigma_{ess}(-L) \subset \left[\frac{1}{r_0}, \infty\right)$ if and only if*

$$\mu(f^2) \leq r\mathscr{E}(f,f) + \beta(r)\mu(|\phi f|)^2, \quad r > r_0, f \in \mathscr{D}(\mathscr{E})$$

*holds for some $\phi \in L^2(\mu)$ and some (decreasing) function $\beta : (r_0, \infty) \to (0, \infty)$.*

Consequently, $\sigma_{ess}(L) = \emptyset$ (i.e. $L$ has purely discrete spectrum) if and only if the intrinsic super Poincaré inequality

$$\mu(f^2) \leq r\mathscr{E}(f,f) + \beta(r)\mu(|\phi f|)^2, \quad r > 0, f \in \mathscr{D}(\mathscr{E}) \qquad (1.1)$$

holds for some $\phi \in L^2(\mu)$ and some (decreasing) function $\beta : (0, \infty) \to (0, \infty)$.

# Functional Inequalities for Discrete Spectrum



We now consider the following Shrördinger operator

$$L_V := L_0 - V,$$

where

- $L_0 \le 0$ is a self-adjoint operator on $L^2(\mu)$ such that $P_t^0 := e^{tL_0}$ is ultracontarctive, i.e.

$$\|P_t^0\|_{L^1(\mu)\to L^\infty(\mu)} < \infty, \quad t > 0.$$

  In this case $P_t^0$ has a bounded heat kernel $p_t^0(x, y)$.

- $V : E \to [0, \infty)$ is measurable such that $\mathscr{D} := \{f \in \mathscr{D}(L_0) : \mu(V f^2) < \infty\}$ is dense in $L^2(\mu)$.

Let $(L_V, \mathscr{D}(L_V))$ be the Friedrichs extension of $(L_V, \mathscr{D})$.

# Functional Inequalities for Discrete Spectrum



**Theorem (W/Wu 08'BMS)**

*If* $\mu(V \le r) < \infty$ *for* $r > 0$, *then* $\sigma_{ess}(L_V) = \emptyset$.

Proof: Let $\mathscr{E}_V(f, f) = -\mu(f L_V g),\ f \in \mathscr{D}(L_V)$. Then $\mu(V \le r) < \infty$ for $r > 0$ implies the intrinsic super Poincaré inequality

$$\mu(f^2) \le r\mathscr{E}_V(f, f) + \beta(r)\mu(|\phi f|)^2, \quad r > 0, f \in \mathscr{D}(L_V) \quad (1.2)$$

for some $\phi \in L^2(\mu)$ and decreasing $\beta : (0, \infty) \to (0, \infty)$. Then Theorem [02'JFA] ensures $\sigma_{ess}(L_V) = \emptyset$. □

# Lower bound for eigenvalues



This result was reproved for $L_0 = \Delta$ on $\mathbb{R}^d$ by B. Simon:

- Schrödinger operators with purely discrete spectrum, Mech. Funct. Anal. Top. 15(2009).
- In introduction he wrote: I was struck by the following simple and elegant theorem... I learned of this result from Wang-Wu [25].

Let $\{\lambda_n\}_{n\geq 1}$ be all eigenvalues listed in increasing order counting multiplicities.

We are able to estimate $\lambda_n$ using the rate function $\beta$ in (1.2).

# Lower bound for eigenvalues

## Theorem (W02'JFA)

*Let $\phi \in L^2(E)$ be positive such that $P_{t_0}\phi \le \mathrm{e}^{\lambda t}\phi$ holds for some $\lambda > 0$ and all $t \ge 0$. If (1.2) holds for some $\beta$ with*

$$\beta(\infty) := \lim_{s\to\infty} \beta(s) = 0,$$

$$\Lambda(t) := \int_t^\infty \frac{\beta^{-1}(r)}{r}\mathrm{d}r < \infty, \quad t > 0,$$

*then for any $\varepsilon \in (0,1)$, there exists a constant $c > 0$ such that*

$$\lambda_n \ge \frac{c}{\Lambda(\varepsilon n)}, \quad n \ge 1. \tag{1.3}$$

In applications, we take as large as possible $\phi \in L^2(\mu)$ such that $\beta$ is as small as possible.

# Non-local Schrödinger operators

Let $J : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$ be measurable such that

$(A)$ $J(x,y) = J(y,x)$ and

$$\sup_{x\in\mathbb{R}^d} \int_{\{|z|\le 1\}} |z|\cdot|J(x,x+z)-J(x,x-z)|\mathrm{d}z < \infty, \quad (2.1)$$

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} (|z|^2 \wedge 1) J(x,x+z)\mathrm{d}z < \infty. \qquad (2.2)$$

Example (stable like): for some constants $0 < \alpha_1 \le \alpha_2 < 2$, $0 < c_1 < c_2 < \infty$ and $\kappa > 0$,

$$c_1 1_{\{|x-y|\le\kappa\}} |x-y|^{-(d+\alpha_1)} \le J(x,y) \le c_2 |x-y|^{-(d+\alpha_2)}.$$

# Non-local Schrödinger operators



Consider the non-local operator

$$L_0 f(x) := \int_{\mathbb{R}^d} \big\{ f(x+z) - f(x) - \langle \nabla f(x), z \rangle 1_{\{|z| \le 1\}} \big\} J(x, x+z) \mathrm{d}z + \frac{1}{2} \int_{\{|z| \le 1\}} \langle \nabla f(x), z \rangle \big( J(x, x+z) - J(x, x-z) \big) \mathrm{d}z, \quad x \in \mathbb{R}^d,$$

which is well defined for $f \in C_0^2(\mathbb{R}^d)$, and is symmetric in $L^2(\mathbb{R}^d)$, hence has Friedrichs extension $(L_0, \mathscr{D}(L_0))$.

# Non-local Schrödinger operators

Let $V \geq 0$ be a locally integrable function on $\mathbb{R}^d$. Then the energy form of the Schrödinger operator $L_V := L_0 - V$ is well defined on $C_0^2(\mathbb{R}^d)$ and has Friedrichs extension $(L_V, \mathscr{D}(L_V))$. The associated energy form is

$$\mathscr{E}_V(f,g) := \int_{\mathbb{R}^d \times \mathbb{R}^d} \big(f(x) - f(y)\big)\big(g(x) - g(y)\big) J(x,y) \mathrm{d}x \mathrm{d}y + \int_{\mathbb{R}^d} (Vfg)(x) \mathrm{d}x, \quad f, g \in \mathscr{D}((-L_V)^{\frac{1}{2}}).$$

We assume that

$$\Phi(R) := \inf_{|x| \geq R} V(x) \uparrow \infty \text{ as } R \uparrow \infty.$$

Then $\mu(V \leq r) < \infty$ for the Lebsesgue measure $\mu$, so that Theorem [W/Wu 08] implies $\sigma_{ess}(L_V) = \emptyset$.

# Lower bound estimate

To derive sharp lower bound of $\lambda_n$, we establish

$$\mu(f^2) \le r\mathscr{E}_V(f,f) + \beta(r)\mu(|f\phi|)^2, \quad r > 0$$

for as small as possible $\beta$. To this end, we take as large as possible $\phi \in L^2(\mu)$ satisfying $P_t^0\phi \le e^{\lambda t}\phi$. Typical choice includes $\phi(x) = \varphi(|x|)$ for

$$\varphi_k(s) := (1+s^2)^{-p}, \quad p > \frac{d}{4},$$

or more generally for $\varphi \in \mathscr{C}$, the class of $\varphi \in C^2([0,\infty))$ such that

# Lower bound estimate

1. $\int_0^\infty \varphi(s)^2 s^{d-1}\mathrm{d}s < \infty.$
2. There exists constant $c > 0$ such that $\sup_{r\le 1+s}\big(|\varphi'(r)|(r+r^{-1}) + |\varphi''(r)|\big) + \varphi(s/2) \le c\varphi(s)$ holds for $s > 0$.

The first condition is equivalent to $\phi := \varphi(|\cdot|) \in L^2(\mu)$, and the second condition implies $P_t^0\phi \le \mathrm{e}^{\lambda t}\phi$ as required in Theorem [W02].

# Lower bound estimate



## Theorem (1)

*Let* $J$ *satisfy* (A), *let* $\Phi(R) := \inf_{|x|\geq R} V(x) \uparrow \infty$. *Assume*

$$J(x,y) \geq \frac{c_1}{|z|^{d+\alpha_1}} 1_{\{|x-y|\leq \kappa\}}, \quad x,y \in \mathbb{R}^d \tag{2.3}$$

*for some constants* $c_1 > 0$ *and* $\alpha_1 \in (0,2)$. *For* $\varphi \in \mathscr{C}$, *let*

$$\Gamma(r) = \inf\left\{s > 0 : s^{\frac{d}{\alpha_1}} \varphi\big(\kappa + \Phi^{-1}(2s^{-1})\big)^2 \geq r^{-1}\right\}, \quad r > 0.$$

*If*

$$\lambda(t) := \int_t^\infty \frac{\Gamma(r)}{r} \mathrm{d}r < \infty, \quad t > 0,$$

*then there exist constants* $\delta_1, \delta_2 > 0$ *such that*

$$\lambda_n \geq \frac{\delta_1}{\lambda(\delta_2 n)}, \quad n \geq 1. \tag{2.4}$$

## Outline of proof



- For $\varphi \in \mathscr{C}$, the function $\phi := \varphi(|\cdot|) \in L^2(\mu)$ and $L_0\phi \leq \lambda\phi$ holds for some constant $\lambda > 0$. So, $P_t^0\phi \leq \phi e^{\lambda t}$ as required in Theorem (W02'JFA).
- According to [X. Chen/J. Wang 16'JFA], there exists a constant $c > 0$ such that for any $R > 0$, we have the local super Poincaré inequality

$$\int_{B(0,R)} f^2(x)\mathrm{d}x \leq r\mathscr{E}_0(f,f) + \frac{c(1+r^{-\frac{d}{\alpha_1}})}{\varphi(R+1)^2}\left(\int_{B(0,R+1)}(|f|\phi)(x)\mathrm{d}x\right)^2, \quad r > 0.$$

# Outline of proof



Combining this with

$$\int_{B(0,R)^c} f^2(x)\mathrm{d}x \le \frac{1}{\Phi(R)} \int_{B(0,R)^c} (Vf^2)(x)\mathrm{d}x$$
$$\le \frac{1}{\Phi(R)}\mu(Vf^2),$$

we obtain

$$\mu(f^2) \le (r \vee \Phi(R)^{-1})\mathscr{E}_V(f,f)$$
$$+ \frac{c(1+r^{-\frac{d}{\alpha_1}})}{\varphi(R+1)^2}\bigg(\int_{B(0,R+1)} (|f|\phi)(x)\mathrm{d}x\bigg)^2, \quad r>0.$$

Taking $R = \Phi^{-1}(r^{-1})$ we prove (1.2) for

$$\beta(r) = c' r^{-\frac{d}{\alpha_1}} \varphi\big(1+\Phi^{-1}(r^{-1})\big)^{-2}, \quad r>0.$$

Then the theorem follows from Theorem (W02'JFA). □

# Consequence

## Corollary

*If* $J(x,y) \ge c_1|x-y|^{-(d+\alpha_1)}1_{\{|x-y|\le\kappa\}}$ *and* $V(x) \ge c|x|^{\theta_1}$ *for some constants* $c_1, \kappa, c, \theta_1 > 0, \alpha_1 \in (0,\alpha_1)$ *and larger* $|x|$. *Then for any* $p > 1$ *there exists a constant* $\delta > 0$ *such that*

$$\lambda_n \ge \delta n^{\frac{\theta_1\alpha_1}{pd(\theta_1+\alpha_1)}}, \quad n \ge 1.$$

*On the other hand, if for some constants* $C, \theta_2 > 0$ *and* $\alpha_2 \in (0,2)$ *such that*

$$J(x,y) \le \frac{C}{|x-y|^{d+\alpha_2}}, \quad x,y \in \mathbb{R}^d,$$

*and* $V(x) \le C(1+|x|^{\theta_2})$, *then there exists a constant* $\delta' > 0$ *such that*

$$\lambda_n \le \delta' n^{\frac{\alpha_2\theta_2}{d(\alpha_2+\theta_2)}}, \quad n \ge 1.$$

# Proof for lower bound

Simply apply Theorem (1) with $\varphi(s) = (1+s^2)^{-\frac{pd}{4}}$. □

We write $f \asymp g$ for two nonnegative functions $f, g$, if there exists a constant $C > 1$ such that

$$\frac{f}{C} \le g \le Cf.$$

Question: Prove the lower bound estimate with $p = 1$, i.e.

$$\lambda_n \ge \delta n^{\frac{\theta_1 \alpha_1}{d(\theta_1+\alpha_1)}}, \quad n \ge 1.$$

So that when

$$J(x,y) \asymp |x-y|^{-(d+\alpha)}, \quad V(x) \asymp |x|^{\theta}$$

we have

$$\lambda_n \asymp n^{\frac{\alpha\theta}{d(\alpha+\theta)}}.$$

# Proof for lower bound

This is true when $\alpha = \theta = 2$ for which the Schrödinger operator reduces to $\Delta - |x|^2$. In this case, the eigenvalues are comparable with those of the O-U operator

$$\Delta - x \cdot \nabla,$$

and thus,

$$\lambda_n \asymp n^{\frac{1}{d}}.$$

In general, the question is not yet fixed.

# Proof for upper bound



- Variational formula of $\lambda_n$:

$$\lambda_n = \inf_{(u_1,\cdots,u_n)\in\mathscr{S}_n} \sup_{u\in B(u_1,\cdots,u_n)} \mathscr{E}_V(u,u), \quad n\geq 1,$$

where $(u_1,\cdots,u_n)\in\mathscr{S}_n$ means

$$u_i\in\mathscr{D}(\mathscr{E}_V), \quad \int_{\mathbb{R}^d}(u_iu_j)(x)\mathrm{d}x = 1_{\{i=j\}}, \quad 1\leq i,j\leq n,$$

and

$$B(u_1,\cdots,u_n) := \Big\{\sum_{i=1}^n a_iu_i : \sum_{i=1}^n a_i^2 = 1\Big\}.$$

# Proof for upper bound



Let

$$\xi(k) = k^{\frac{\alpha_2}{\theta_2+\alpha_2}}, \quad k \ge 1,$$

$$h_k(s) = \min\left\{(s-\xi(k))^+, (\xi(k+1)-s)^+\right\}, \quad s \in \mathbb{R}, k \ge 1.$$

Let

$$G_n = \{1, 2, \cdots, n\}^d = \left\{(k_1, \cdots, k_d) : \ 1 \le k_i \le n, 1 \le i \le d\right\},$$

$$u_{\mathbf{k}}(x) := \prod_{i=1}^d h_{k_i}(x_i), \quad \mathbf{k} := (k_1, \cdots, k_d) \in G_n.$$

Since $\{u_{\mathbf{k}}\}_{\mathbf{k}\in G_n}$ have disjoint supports, they are orthogonal each other. It is easy to see that

$$I_{\mathbf{k}} := \int_{\mathbb{R}^d} u_{\mathbf{k}}(x)^2 \mathrm{d}x \asymp \prod_{i=1}^d k_i^{-\frac{3\theta_2}{\alpha_2+\theta_2}}, \quad \mathbf{k} = (k_1, \cdots, k_d) \in G_n.$$

# Proof for upper bound



Let

$$\tilde{u}_{\mathbf{k}} = \frac{u_{\mathbf{k}}}{\sqrt{I_{\mathbf{k}}}}, \quad \mathbf{k} \in G_n.$$

Since $\# G_n = n^d$, the variational formula gives

$$\lambda_{n^d} \le \sup_{u \in B(\tilde{u}_{\mathbf{k}}:\ \mathbf{k} \in G_n)} \mathscr{E}_V(u,u), \quad n \ge 1.$$

- We have $\mathscr{E}_V(\tilde{u}_{\mathbf{k}}, \tilde{u}_{\mathbf{k}}) \le c_1 n^{\frac{\theta_2 \alpha_2}{\theta_2 + \alpha_2}}, \quad n \ge 1, \mathbf{k} \in G_n.$
- For $u := \sum_{\mathbf{k} \in G_n} a_{\mathbf{k}} \tilde{u}_{\mathbf{k}}$ with $\sum_{\mathbf{k} \in G_n} a_{\mathbf{k}}^2 = 1$, we intend to prove the same upper bound estimate for

$$\mathscr{E}_0(u,u) = \int_{\mathbb{R}^d \times \mathbb{R}^d} |u(x+z) - u(x)|^2 J(x, x+z) \mathrm{d}x \mathrm{d}z$$
$$= \sum_{\mathbf{k}, \mathbf{k}' \in G_n} a_{\mathbf{k}} a_{\mathbf{k}'} \mathscr{E}_0(u_{\mathbf{k}}, u_{\mathbf{k}'}).$$

# Proof for upper bound

Obviously,

$$\mathscr{E}_0(u,u) \le \sum_{\mathbf{k},\mathbf{k}'\in G_n} \Big| a_{\mathbf{k}} a_{\mathbf{k}'} \mathscr{E}_0(u_{\mathbf{k}}, u_{\mathbf{k}'}) \Big|.$$

This together with the upper bound of $\mathscr{E}_0(u_{\mathbf{k}}, u_{\mathbf{k}})$ would imply the worse upper bound

$$\mathscr{E}_0(u,u) \le c n^{\frac{\theta_2\alpha_2}{\theta_2+\alpha_2}} \asymp n^{\frac{1}{d}+\frac{\theta_2\alpha_2}{\theta_2+\alpha_2}}.$$

This is however less sharp! To derive better upper bound, we see that for $\mathbf{k},\mathbf{k}'\in G_n$ with $\|\mathbf{k}-\mathbf{k}'\|_\infty := \max |k_i - k_i'| \ge 2$,

$$|u_{\mathbf{k}}(x+z) - u_{\mathbf{k}}(x)| \cdot |u_{\mathbf{k}'}(x+z) - u_{\mathbf{k}'}(x)| \ne 0$$

implies

$$|z| \ge c_0 n^{-\frac{\theta_2}{\theta_2+\alpha_2}}.$$

# Proof for upper bound



So,

$$\begin{aligned}
\mathscr{E}_0(u,u) &= \int_{\mathbb{R}^d\times\mathbb{R}^d} |u(x+z)-u(x)|^2 J(x,x+z)\mathrm{d}x\mathrm{d}z \\
&\le 2c_2 \int_{\mathbb{R}^d\times\mathbb{R}^d} 1_{\{|z|\ge c_0 n^{-\frac{\theta_2}{\theta_2+\alpha_2}}\}} \frac{u(x+z)^2+u(x)^2}{|z|^{d+\alpha_2}}\mathrm{d}x\mathrm{d}z \\
&+ \sum_{\|\mathbf{k}-\mathbf{k}'\|_\infty\le 1} \left| a_{\mathbf{k}} a_{\mathbf{k}'} \sqrt{\mathscr{E}_0(\tilde u_{\mathbf{k}},\tilde u_{\mathbf{k}})\mathscr{E}_0(\tilde u_{\mathbf{k}'},\tilde u_{\mathbf{k}'})} \right| \\
&\le 4c_2 \int_{\{|z|\ge c_0 n^{-\frac{\theta_2}{\theta_2+\alpha_2}}\}} \frac{\mathrm{d}z}{|z|^{d+\alpha_2}} \\
&+ \frac{1}{2} \sum_{\|\mathbf{k}-\mathbf{k}'\|_\infty\le 1} \left\{ a_{\mathbf{k}}^2 \mathscr{E}_0(\tilde u_{\mathbf{k}},\tilde u_{\mathbf{k}}) + a_{\mathbf{k}'}^2 \mathscr{E}_0(\tilde u_{\mathbf{k}'},\tilde u_{\mathbf{k}'}) \right\} \\
&\le c_3 n^{\frac{\theta_2\alpha_2}{\theta_2+\alpha_2}}
\end{aligned}$$

# Proof for upper bound

Finally, since when $\mathbf{k} \neq \mathbf{k}'$ the functions $u_{\mathbf{k}}$ and $u_{\mathbf{k}'}$ have disjoint support, we have

$$\int_{\mathbb{R}^d} (u^2 V)(x)\mathrm{d}x = \sum_{\mathbf{k}\in G_n} a_k^2 \int_{\mathbb{R}^d} (u_{\mathbf{k}}^2 V)(x)\mathrm{d}x \le c_4 n^{\frac{\theta_2\alpha_2}{\theta_2+\alpha_2}}.$$

In conclusion,

$$\lambda_{n^d} \le \sup_{u\in B(\tilde{u}_{\mathbf{k}}:\ \mathbf{k}\in G_n)} \mathscr{E}_V(u,u) \le c_5 n^{\frac{\theta_2\alpha_2}{\theta_2+\alpha_2}}, \quad n \ge 1.$$

In general, for any $n \ge 1$, let $r_n = \inf\{k \in \mathbb{Z}_+ : k \ge n^{\frac{1}{d}}\}$. We have

$$\lambda_n \le \lambda_{(1+r_n)^d} \le c_5(1+r_n)^{\frac{\theta_2\alpha_2}{\theta_2+\alpha_2}} \le \delta n^{\frac{\theta_2\alpha_2}{d(\theta_2+\alpha_2)}}, \quad n \ge 1.$$

# Example



Consider the following stable-like jump kernel

$$J(x,y) = \frac{n(x,y)}{|x-y|^{d+\alpha(x,y)}} 1_{\{|x-y| \le \kappa\}} + q(x,y), \quad x,y \in \mathbb{R}^d,$$

where $\kappa > 0$ is a constant and

$n, \alpha, q : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ are measurable and symmetric, such that

(a) $\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} q(x,y) \mathrm{d}y < \infty$;

(b) There exists a constant $\varepsilon \in (0,1)$ such that $\varepsilon \le n(x,y) \le \varepsilon^{-1}, x, y \in \mathbb{R}^d$;

(c) There exists constants $2 > \alpha_2 \ge \alpha_1 > 0$ such that $\alpha_1 \le \alpha(x,y) \le \alpha_2, x, y \in \mathbb{R}^d$;

(d) $\sup_{x \in \mathbb{R}^d} \int_{\{|z| \le 1\}} \frac{|n(x,x+z) - n(x,x-z)|}{|z|^{d+\alpha_2 - 1}} \mathrm{d}z < \infty$.

# Example



If there exist constants $C > 1, \theta_1, \theta_2 > 0$ such that

$$q(x,y) \le \frac{c}{|x-y|^{d+\alpha_2}}, \quad x, y \in \mathbb{R}^d,$$
$$\frac{|x|^{\theta_1}}{C} \le V(x) \le C|x|^{\theta_2}, \quad |x|\rangle 1,$$

then for any $p > 1$

$$\delta_1 n^{\frac{\theta_1 \alpha_1}{pd(\theta_1+\alpha_1)}} \le \lambda_n \delta' n^{\frac{\theta_2 \alpha_2}{d(\theta_2+\alpha_2)}}.$$

If in particular $\alpha_1 = \alpha_2 = \alpha, \theta_1 = \theta_2 = \theta$, then

$$\lim_{n\to\infty} \frac{\log \lambda_n}{\log n} = \frac{\theta\alpha}{d(\theta+\alpha)}.$$

# The local case



Consider the following second order differential operator on $\mathbb{R}^d$:

$$L_{a,V} = \mathrm{div}(a\nabla) - V,$$

where $V$ is a locally integrable nonnegative function on $\mathbb{R}^d$, and $a \in C_b^2(\mathbb{R}^d; \mathbb{R}^d \otimes \mathbb{R}^d)$ satisfying the uniform ellipticity condition

$$c|z|^2 \le \langle a(x)z, z\rangle, \quad x, z \in \mathbb{R}^d$$

for some constant $c > 0$. Then $(L_{a,V}, C_0^\infty(\mathbb{R}^d))$ is a negative definite linear operator in $L^2(\mathbb{R}^d)$. Let $(L_{a,V}, \mathscr{D}(L_{a,V}))$ be the Friedrichs extension. When $\mathrm{vol}(V \le r) < \infty$ for $r > 0$, the essential spectrum of $L_{a,V}$ is empty.

# The local case

## Theorem

*For any* $\varphi \in \mathscr{C}$, *let*

$$\Gamma(r) = \inf\left\{s > 0 : s^{\frac{d}{2}}\varphi\big(1 + \Phi^{-1}(s^{-1})\big)^2 \geq r^{-1}\right\}, \quad r > 0.$$

*If*

$$\lambda(t) := \int_t^\infty \frac{\Gamma(r)}{r}\mathrm{d}r < \infty, \quad t > 0,$$

*then there exist constants* $\delta_1, \delta_2 > 0$ *such that*

$$\lambda_n \geq \frac{\delta_1}{\lambda(\delta_2 n)}, \quad n \geq 1.$$

# The local case



If, in particular, $V(x) \geq c|x|^\theta$ for some constants $c, \theta > 0$ and large $|x|$, then for any $k \in \mathbb{Z}_+$ and $p > 1$, there exists a constant $\delta > 0$ such that

$$\lambda_n \geq \Big(\frac{\delta n}{\{\log^{\otimes(k+1)}(\mathrm{e}^k + n)\}^p \prod_{i=1}^k \log(\mathrm{e}^i + n)}\Big)^{\frac{2\theta}{d(\theta+2)}}, \quad n \geq 1.$$

On the other hand, if $V(x) \leq c'|x|^{\theta'}$ for some constants $c', \theta' > 0$ and large $|x|$, then there exists a constant $\delta' > 0$ such that

$$\lambda_n \leq \delta' n^{\frac{2\theta'}{d(\theta'+2)}}, \quad n \geq 1.$$

# The local case



When $V(x) \asymp |x|^2$ the operator is comparable with $\Delta - x \cdot \nabla$, so that

$$\delta n^{\frac{1}{d}} \le \lambda_n \le \delta' n^{\frac{1}{d}}, \quad n \ge 1.$$

In general, for $V(x) \asymp |x|^\theta$ we have

$$\lim_{n\to\infty} \frac{\log \lambda_n}{\log n} = \frac{2\theta}{d(\theta+2)}.$$

*Thank You*